Once upon time, there were k lattice points inside a polytope...

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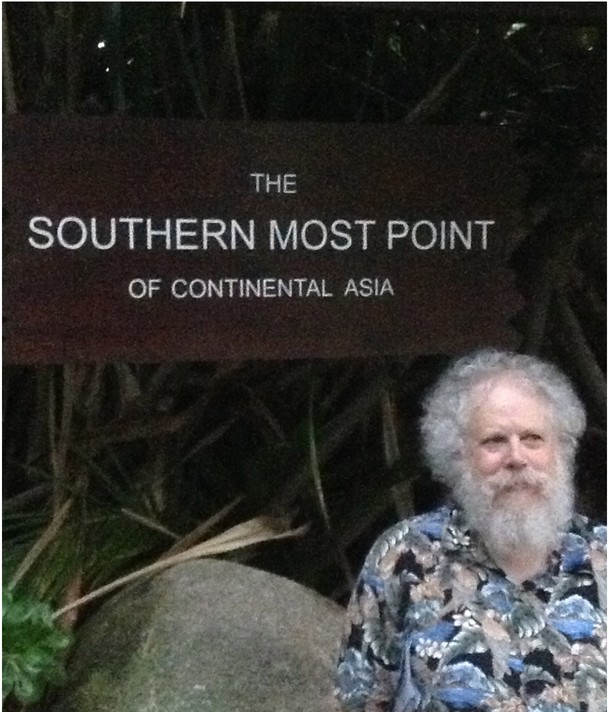
Bruce Reznick's Fest
Bern 2019


THE
SOUTHERN MOST POINT
OF CONTINENTAL ASIA

Bruce like's polytopes too!!

CLEAN LATTICE TETRAHEDRA

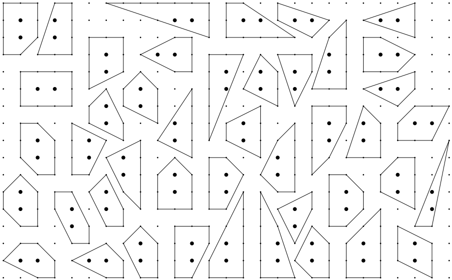
BRUCE REZNICK

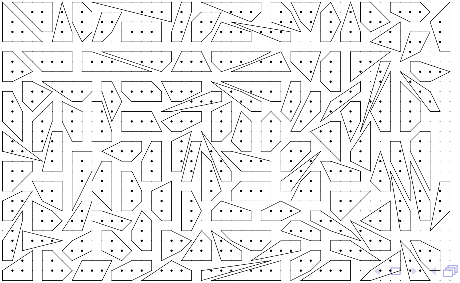
ABSTRACT. A clean lattice tetrahedron is a non-degenerate tetrahedron with the property that the only lattice points on its boundary are its vertices. We present some new proofs of old results and some new results on clean lattice tetrahedra, with an emphasis on counting the number of its interior lattice points and on computing its lattice width.

1. INTRODUCTION AND OVERVIEW

Let $T = T(v_1, \dots, v_n) = conv(v_1, \dots, v_n)$ be a non-degenerate simplex with vertices $v_j \in \mathbb{Z}^n$. We say that T is *clean* if there are no non-vertex lattice points on the boundary of T. Let $i(T) = \#\{int(T) \cap \mathbb{Z}^n\}$ denote the number of lattice points in the interior of a clean lattice simplex T. If $i(T) = k$, then T is called a *k-point lattice simplex*. If $i(T) = 0$, then T is called *empty*. This paper is mainly concerned with clean tetrahedra.

A k-lattice polytope is a lattice polytope containing exactly k lattice points in its interior. Here are all 2-lattice gons and 3-lattice gons.





A question around Bruce's flavor

Given m Linear Inequalities with rational coefficients, they define a polytope.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_n \leq b_m$$

If we want exactly k-lattice points inside, what is the smallest number of constraints m one really needs?
Let us start with $k = 0$. First, I want not a single lattice point inside!
Clearly if $m = 1, 2$ that is not enough!!
Polyhedra with no interior integral points are called lattice-free. The interest in lattice-free polyhedra is motivated by applications in mixed-integer optimization.

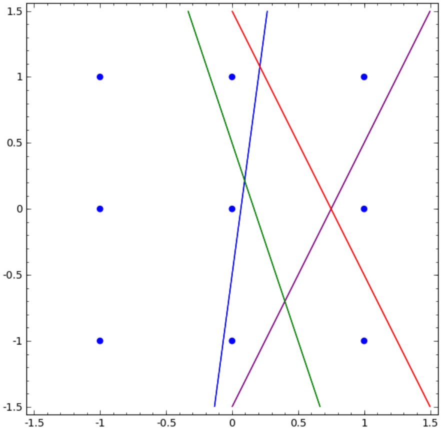
Jean-Paul Doignon, David E. Bell & Herbert Scarf (1970's)



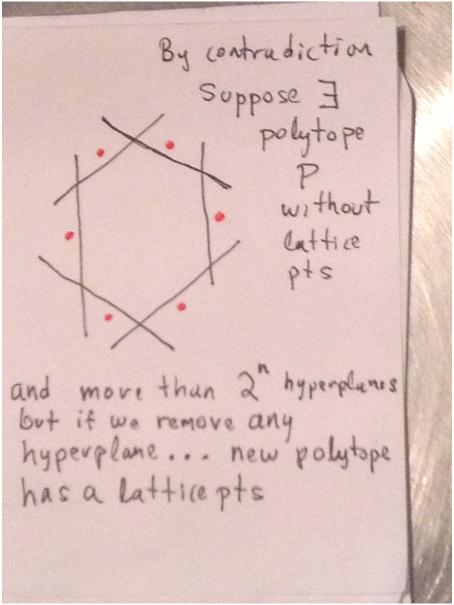
They found the answer...

Theorem of Doignon-Bell-Scarf

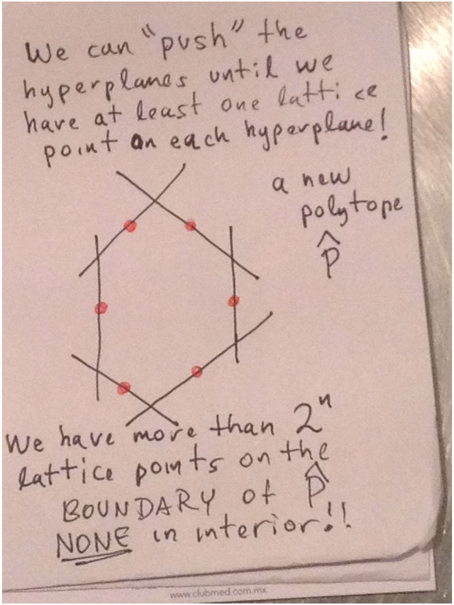
- **Theorem** Let A be a $m \times n$ matrix and b a vector of $\mathbf{Q}^m$. If the problem $P_A(b) = \{x : Ax \leq b, x \in \mathbf{Z}^n\}$ has no integer solution, then there is a subset S of the m rows of A of cardinality no more than 2^n, so that the smaller system has no integer solution either.



A proof from the book 1



A proof from the book 2



A proof from the book 3

Each lattice point has
a pattern of parities
(Even, odd, odd, even, ..., odd)

⇒ A vector of parities
is repeated!!

⇒ The midpoint of those
two red points
is a lattice point
in the INTERIOR
of $\hat{P}$
a contradiction!

Next case...

If one wants to have exactly $k \geq 1$ integral points inside the polytope,

how many hyperplanes does one need to enclose them ??

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There exists a magic number $c(k, n)$, depending only on k and n, such that If the polytope $P_A(b) = \{x : Ax \leq b\}$ has exactly k integral solutions, then a subset of the inequalities of $Ax \leq b$, of cardinality no more than $c(k, n)$, has exactly the same k integer solutions as $P_A(b)$.

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ORIGINAL DOIGNON-BELL-SCARF is case of $k = 0$.

Corollary

For n, k non-negative integers, there exists a magic number $c(k, n)$, determined by k and n, such that

- For any system of inequalities $\{x : Ax \leq b\}$ in $\mathbf{R}^n$, if every subset of the constraints of cardinality $c(k, n)$ has at least k integer solutions, then the entire system of inequalities must have at least k integral solutions.

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- Let $(X_i)_{i \in \Lambda}$ be a collection of convex sets in $\mathbf{R}^n$, where at least one of these sets is compact.
  If exactly k integer points are in $\bigcap_{i \in \Lambda} X_i$, then there is a subcollection of size less than or equal to $c(n, k)$ with exactly the same integer points in their intersection.

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But, WHAT IS THE MAGIC NUMBER $c(n,k)$?

Well, when $k=0$ we knew $c(0,n)=2^n$.

MAIN THEOREM 2: Bound for $c(n, k)$

- **Theorem 2** For n, k non-negative integers

$$c(k, n) \leq \lceil 2(k+1)/3 \rceil 2^n - 2\lceil 2(k+1)/3 \rceil + 2$$

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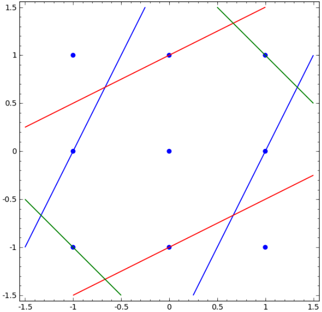
- **Example** For $c(1,2) = 6$ and $c(1,3) = 14$, but $c(3,2) = 6$ (but 8 is our bound!)

- OPEN PROBLEM Find the exact value of the $c(k, n)$ for $k \geq 2$.




Happy Birthday