Ask Bruce for a Form a Day keeps the Doctor Away

Ask Bruce for a Form a Day keeps the Doctor Away or Analogues of Hilbert's 1888 for Symmetric (LAA 2016) and Even Symmetric Forms (JPAA 2017)

Charu Goel (Nagpur India), Salma Kuhlmann (Universität Konstanz),
Bruce Reznick (Urbana Champagne)

Brucefest@66

July 8th, 2019

Plan of the talk

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1. Preliminaries and Hilbert's 17th Problem

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2. Hilbert's 1888 Theorem

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3. Hilbert's 1888 Theorem for Symmetric Forms

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4. Hilbert's 1888 Theorem for Even Symmetric Forms

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- But what if rational functions are not allowed in the sos representation and we want only sos of polynomials?

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- $(\mathcal{Q})$: For what pairs $(n, 2d)$ we have $\mathcal{P}_{n,2d} \subseteq \Sigma_{n,2d}$?

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For the only if direction, Hilbert proved that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, and observed:

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- **Proposition [BCR]:** Let R be a real closed field and p an irreducible polynomial in $R[x_1, \ldots, x_n]$. TFAE:
  1. $(p) = \mathcal{I}(Z(p))$, where $\mathcal{I}(A) = \left\{g \in R[\underline{x}] \mid g(\underline{a}) = 0 \ \ \forall\ \underline{a} \in A\right\}$ is the ideal of vanishing polynomials on $A \subseteq R^n$ and $Z(p) = \left\{\underline{x} \in R^n \mid p(\underline{x}) = 0\right\}$ is the zero set of p.
  2. The sign of the polynomial p changes on R^n (i.e. $p(x)p(y) < 0$ for some $x, y \in R^n$).

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  - We will construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$

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- **Timofte's Half Degree Principle for Symmetric Polynomials :** A symmetric real polynomial of degree $2d$ in n variables is nonnegative (> 0 respectively) on $\mathbb{R}^n \Leftrightarrow$ it is nonnegative (> 0 respectively) on the subset $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid$ number of distinct components in $\underline{x}$ is $\leq k\}$, where $k := \max\{2, d\}$.

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so, $L_n(\underline{x}) = m(n-m)k(n-k)(r-s)^4 - [k(n-k)(r-s)^2]^2$

$= k(n-k)(r-s)^4[m(n-m) - k(n-k)]$

$= k(n-k)(r-s)^4[(m-k)(n-m-k)] \geq 0.$ □

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  - $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = \underbrace{(n, 6)_{n\geq 3}}_{\text{(C-L-R)}}, \underbrace{(3, 10), (4, 8)}_{\text{(Harris)}}$.

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  - construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n \geq 5}$
  - deduce that for $(n, 2d) = (n, 6)_{n \geq 3}$, $(n, 8)_{n \geq 4}, (3, 2d)_{d \geq 5}$, $(n, 2d)_{n \geq 4, d \geq 7}$, the answer to $\mathcal{Q}(S^e)$ is negative.

4.1. Degree jumping principle

- **Lemma 4.1:** If $2t = 4, 6$, and $n \geq 3$, then

$$h_t(x_1, \dots, x_n) := \sum_{i=1}^{n} x_i^{2t} - 10 \sum_{i \neq j} x_i^{2t-2} x_j^2$$

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- **Theorem 4.2** [Degree jumping principle]:
Let $n \geq 3$. If $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$, then
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  2. $(x_1 \ldots x_n)^2 f \in S\mathcal{P}^e_{n,2d+2n} \setminus S\Sigma^e_{n,2d+2n}$.

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  then the complete answer to $\mathcal{Q}(S^e)$ will be:

  $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$ if and only if $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$.

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  - **Theorem:** The form

$$B(x_1, \ldots, x_5) := L_5(x_1^2, \ldots, x_5^2) \in S\mathcal{P}^e_{5,8} \setminus S\Sigma^e_{5,8},$$

(recall that $L_{2m+1} = m(m+1)\sum_{i<j}(x_i - x_j)^4 - \left(\sum_{i<j}(x_i - x_j)^2\right)^2$ is a symmetric psd not sos $(2m+1)-$ary quartic form).

4.2.1. Psd not sos even symmetric $n-$ary octics for $n \geq 6$

- **Theorem:** For $m \geq 3$,
    1. $M_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in S\mathcal{P}^e_{2m+1,8} \setminus S\Sigma^e_{2m+1,8}$, and
    2. $D_{2m} := C_{2m}(x_1^2, \ldots, x_{2m}^2) \in S\mathcal{P}^e_{2m,8} \setminus S\Sigma^e_{2m,8}$,

Set $M_r(x_1, \cdots, x_n) := x_1^r + \cdots x_n^r$. Use it to construct psd not sos even symmetric n-ary dedics and dodedics.

4.3. Hilbert's 1888 Theorem for Even Symmetric forms

Theorem:

1. $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ iff $n = 2, d = 1, (n, 2d) = (n, 4)_{n \geq 3}$, $(3, 8)$.

i.e.

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	×	×	...
10	✓	×	×	×	×	×
12	✓	×	×	×	×	×
14	✓	×	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

THANKS BRUCE !