On an Unpublished Article by Bruce Reznick

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Conference in Honor of Bruce Reznick

Things in Common

The Paper in the Title

"On Hilbert's Construction of Positive Polynomials"

Timeline:

Uploaded to Arxiv on July 14, 2007.

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May 9-11, 2009 FRG Meeting at MIT organized by Pablo Parrilo

Bruce's Paper

Clear explanation and generalization of Hilbert's Method.

Take two cubics F and G intersecting transversely in 9 real points.

Consider 8 of the points. There are $28 - 8 \cdot 3 = 4$ sextics double vanishing at the 8 points.

Therefore there is a sextic R double-vanishing on the 8 points, which does not vanish on the 9-th point (it is not spanned by F^2, G^2 and FG).

Then $F^2 + G^2 + \epsilon R$ will be nonnegative but not a sum of squares.

My Inspiration

Consider the cones $P_{3,6}$ and $\Sigma_{3,6}$. Let $v \in \mathbb{R}^3$ be a point and consider faces $P_{3,6}(v)$ and $\Sigma_{3,6}(v)$ of nonnegative forms and sums of squares vanishing on v.

Then

$$\dim P_{3,6}(v) = \dim P_{3,6} - 3 \quad \text{and} \quad \dim \Sigma_{3,6}(v) = \dim \Sigma_{3,6} - 3.$$

Now consider doing this for 7 points:

$$\dim P'_{3,6} = 28 - 7 \times 3 = 7$$

but sums of squares come from the $10 - 7 = 3$ sextics vanishing on the 7 points, so

$$\dim \Sigma'_{3,6} \leq \binom{3+1}{2} = 6.$$

Switching to Varieties

Instead of forms of arbitrary even degree, we can consider quadratic forms on varieties, but using the Veronese embedding!

Example:

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5 \quad \text{via} \quad \nu_2([x : y : z]) = [x^2 : y^2 : z^2 : xy : xz : yz].$$

Forms of degree $2d$ on X correspond precisely to quadratic forms on $\nu_d(X)$. So we have:

$$P_{X,2d} = P_{\nu_d(X)} \quad \text{and} \quad \Sigma_{X,2d} = P_{\nu_d(X)}.$$

Projecting Away from Points

Let X be a projective variety, and let v be a generic point of X.

Let X_v be the projection away from v.

Key Observation: Σ_{X_v} is the face $\Sigma_X(v)$ of the cone Σ_X, corresponding to forms vanishing on v, and $P_{X_v} \subseteq P_X(v)$.

$$P_{X_v} \subseteq P_X(v) = \Sigma_X(v) = \Sigma_{X_v}.$$

Conclusion: Projection away from a point preserves equality between cones.

We also have

$$\operatorname{codim} X_v = \operatorname{codim} X - 1 \qquad \text{and} \qquad \deg X_v = \deg X - 1.$$

The Finale

Projection away successively from $\operatorname{codim} X - 1$ many generic points, we obtain a hypersurface Y of degree at least 3.

$$\deg X \geq \operatorname{codim} X + 1.$$

And if

$$\deg X > \operatorname{codim} X + 1$$

then $\deg Y \geq 3$.

Since the ideal of Y has no forms of degree 2, elements of Σ_Y are globally nonnegative quadratic forms, while elements of P_Y are quadratic forms nonnegative on Y. Therefore,

$$\Sigma_Y \subsetneq P_Y \qquad \text{and} \qquad \Sigma_X \subsetneq P_X.$$

THANK YOU!