DISCRETE CONES AND CARATHÉODORY'S THEOREM

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In remembrance of my parents

Discrete cones, in systematic terminology: normal affine semigroups, represent the link between branches of mathematics that, at first sight, seem to be extremely distant from one another, i.e. combinatorial geometry and discrete optimisation on the one hand, and algebraic geometry and commutative algebra on the other. These branches and the links between them have been researched intensively in recent decades.

The disproof of two conjectures about the combinatorial structure of discrete cones will be discussed in this paper. In particular, we will show that Carathéodory's theorem for convex cones does not have an integer analogue. This is proved by a discrete cone which was recently found by Joseph Gubeladze (Tbilisi) and the author. However, it was first only recognised as a counterexample to the second conjecture to be discussed—the so-called unimodular Hilbert covering. The fact that it also violates the discrete Carathéodory property was then found out by Martin Henk, Alexander Martin and Robert Weismantel (Magdeburg and Berlin) and represented in a joint paper [4].

The discovery of the counterexamples signifies a success for *experimental mathematics*, since it marks the end of a very lengthy series of experiments. This success is largely due to the capacity of modern computers which enable very large numbers of objects to be searched, even for complex properties.

For numerous other, in particular, algebraic aspects of the theory of discrete cones, we refer to [3] and the joint work [5] of the author with Gubeladze and Ngô Viet Trung (Hanoi). Gubeladze's guest stay at the University of Vechta in 1995, financed by the Deutsche Forschungsgemeinschaft, contributed considerably to the latter work. The paper [3], which in conjunction with [4] forms the basis of this article, came into being during Gubeladze's guest stay (from October 1996 to March 1998) as a research scholar of the Alexander von Humboldt Foundation at the Department of Mathematics/Informatics at the University of Osnabrück.

CONVEX AND DISCRETE CONES

The solutions $x = (\xi_1, \dots, \xi_n)$ of a system

$$a_{i1}\xi_1 + \cdots + a_{in}\xi_n = 0, \qquad i = 1, \dots, m,$$

of m homogeneous linear equations form a linear subspace U of the n-dimensional vector space. (In the following, for the sake of clarity, vectors will be denoted by the letters x, y and z, their components by the corresponding Greek letters and coefficients by a, b and c). This system is solved by determining a basis of U, i.e. vectors $z_1, \dots, z_u \in U$, $u = \dim(U)$, for which every solution x has a (uniquely determined) representation $x =$

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$c_1z_1 + \cdots + c_uz_u$ as a *linear combination* of $z_1, \dots, z_u$. (Moreover, every such linear combination is also the solution of the system of equations). In the following we will first of all discuss whether these statements remain valid or how they should possibly be altered if inequations are to be considered instead of equations. Systems of inequations occur in numerous applications of mathematics, especially in optimisation.

The set of solutions of a system

$$b_{k1}\xi_1 + \cdots + b_{kn}\xi_n \geq 0, \qquad k = 1, \dots, p,$$

of p homogeneous linear inequalities forms a polyhedral convex cone C. Without loss of generality, we can assume that C has the same dimension as its surrounding space and, moreover, does not contain any complete line. The latter property is equivalent to C having a vertex at zero. Since only a finite number of inequalities are considered, the cone has a finite number of *facets*, and a *support hyperplane* passes through each facet.

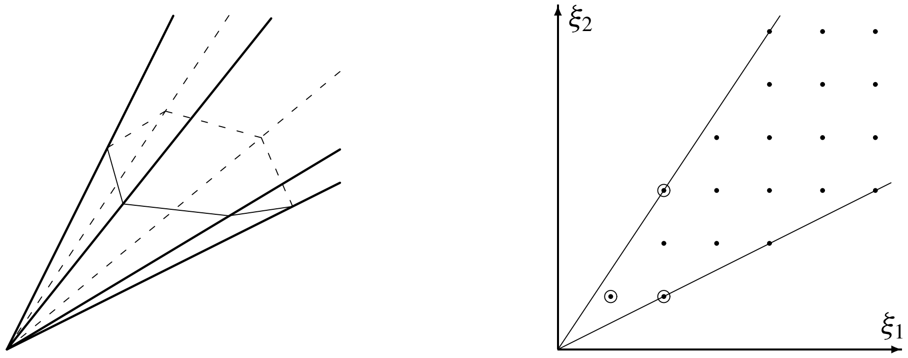

FIGURE 1. A convex and a discrete cone

This explains why the cone is also "finite" in another sense: If we select a point z_j, $j = 1, \dots, u$, on each of the finite number of edges, all points $x \in C$ can be written as a linear combination of $z_1, \dots, z_u$ with *non-negative* coefficients,

$$x = c_1z_1 + \cdots + c_uz_u, \qquad c_j \geq 0, \ j = 1, \dots, u,$$

(conversely, every such linear combination is a solution of the system of inequations). None of the z_i can be omitted; for this reason, $z_1, \dots, z_u$ form a *minimal generating system*. As soon as the cone C has at least dimension 3, it can have any number of edges. Hence in general, there is no generating system with $\dim(C)$ elements. If we consider each point of the cone *individually*, however, we can represent it with $\dim(C)$ selected elements of $z_1, \dots, z_u$. This is

Carathéodory's theorem. *Let C be a convex cone of the dimension n which is generated by $z_1, \dots, z_u$. Then for every $x \in C$ there are indices $i_1, \dots, i_n$ and $a_j \geq 0$ coefficients so that $x = a_1z_{i_1} + \cdots + a_nz_{i_n}$.*

Carathéodory's theorem follows from the stronger statement that a *cross-section* of the cone can be triangulated, as shown in Figure 2 for a 3-dimensional cone. If the ray through x emitting from zero intersects the cross-section in a triangle Δ, we then select

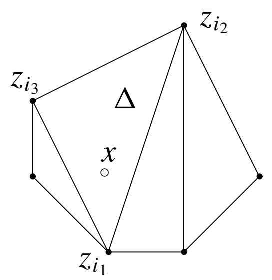

FIGURE 2. Triangulation of the cross-section

those z_i's to represent x that lie on the rays through the boundary point of Δ. (In general, $(n-1)$-dimensional simplices take the place of the triangles).

A *discrete cone* S is the integer analogue of a convex cone: it occurs when one restricts the set of solutions of a system of homogeneous linear inequalities containing *integer coefficients* to vectors with *integer components*. A convex cone C also belongs to each discrete cone S, and S is the set of integer vectors in C. Finally, a discrete cone is also finitely generated:

Gordan's Lemma. *A discrete cone S always contains elements $z_1, \dots, z_w$, for which each element $x \in S$ possesses a representation $x = a_1 z_1 + \dots + a_w z_w$ with integers $a_i \geq 0$.*

Gordan's lemma can still be considerably strengthened: There is even a uniquely defined minimal generating system of S which (in the combinatorial literature) is called the *Hilbert basis* of S, or Hilb(S) for short. Hilb(S) is obtained by selecting a finite generating system, according to Gordan's lemma, and then omitting the superfluous elements. The Hilbert basis, in fact, consists of all *irreducible* vectors y in S, i.e. those vectors that can not be written as $y = y_1 + y_2$ with $y_1, y_2 \in S$, $y_1, y_2 \neq 0$. If a comparison is made to number theory, they are the "prime numbers" in S—however, they occur in a finite number, and the representation of a certain element by them is, in general, not unique. The discrete cone shown in Figure 1 is described by the inequalities $-\xi_1 + 2\,\xi_2 \geq 0$ and $3\,\xi_1 - 2\,\xi_2 \geq 0$. Its Hilbert basis consists of (1, 1), (2, 1) and (2, 3).

Now the question arises whether even an analogy to Carathéodory's theorem is valid for discrete cones:

Discrete Carathéodory property. *Let S be a discrete cone of dimension n. Can $z_1, \dots, z_n$ of* Hilb(S) *and whole numbers $a_1, \dots, a_n \geq 0$ be found for each element x of S elements, for which $x = a_1 z_1 + \dots + a_n z_n$?*

Sebö [11] has proved for $n = 3$ and conjectured for all n that the answer to this question is *yes*. (This is very easy to see for $n = 1$ and $n = 2$). The counterexamples we have found are of dimension 6. The question remains unsolved for dimensions 4 and 5.

UNIMODULAR TRIANGULATIONS AND COVERS

We intend to derive Sebö's theorem for three-dimensional discrete cones S in a special case. The proof leads us to triangulations, as in Carathéodory's theorem, and more generally, to covers of discrete cones. The feature of this special case is that all elements of Hilb(S) lie on one plane. After an integer change of coordinates it can be assumed that this plane E is given by the equation $\zeta_3 = 1$, i.e. that Hilb(S) consists of vectors of the form $(\zeta_1, \zeta_2, 1)$, (this requires an explanation which we, however, will pass over). The left-hand side of Figure 3 shows a "vertical" cross-section of such a cone. The right-hand side of the figure shows cross-section Q along plane E in which the elements of the Hilbert basis are denoted by dots. As shown in the figure, we triangulate Q such that each

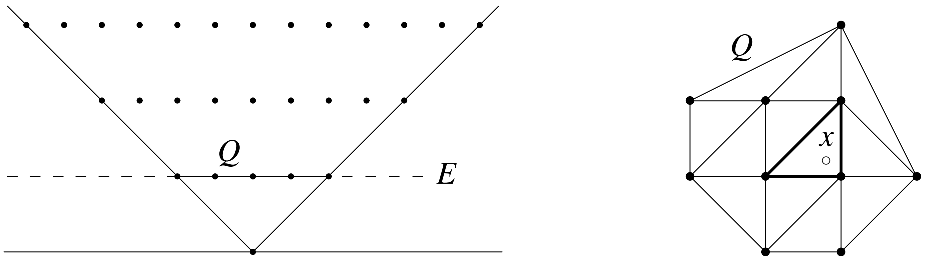

FIGURE 3. Vertical cut and cross-section

triangle Δ contains no other elements of the Hilbert basis apart from its corners z_1, z_2, z_3. According to a classical theorem of elementary integer geometry, Pick's formula, Δ must therefore have area $1/2$. The volume of the pyramid with basis Δ and its vertex at zero is then $1/6$, and the parallelotope spanned by the three vectors z_1, z_2, z_3 has volume 1. But this means

$$\delta = \det(z_1, z_2, z_3) = \pm 1.$$

We now consider a vector x in S. The ray from zero through x intersects one of the triangles Δ; in other words, x lies in cone D spanned by z_1, z_2, z_3. Since $\delta \neq 0$, the system of equations

$$x = c_1 z_1 + c_2 z_2 + c_3 z_3$$

has a unique solution. Since x lies in D, $c_1, c_2, c_3 \geq 0$ is valid, and finally, the c_i's even have to be whole numbers because $\delta = \pm 1$. The latter follows from Cramer's rule, since all coefficients in the system and the right-hand side are integers, and only δ appears as the denominator.

Hence it is shown that three-dimensional cones C in the considered special case possess the discrete Carathéodory property. We have even derived a more precise statement: C possesses a *unimodular Hilbert triangulation*. The prefix "Hilbert" indicates that the corners of the triangles (more generally: the $(n-1)$-simplices) are given by elements of the Hilbert basis, whilst "unimodular" reflects the condition $\delta = \pm 1$.

As shown by Sebö, all three-dimensional discrete cones possess unimodular Hilbert triangulations. This statement was also proven independently by Aguzzoli and Mundici [1] and Bouvier and Gonzalez–Sprinberg [2]. An example of a four-dimensional cone without such a triangulation can also be found in [2].

Even in dimension 3 there is no analogue to Pick's formula: One can easily obtain tetrahedra of arbitrarily large volume whose corners are their only integer points. Although the above procedure can not be transferred naively to higher dimensions, the actual idea has not yet been fully exhausted. We have proceeded further than necessary in one respect: To derive the discrete Carathéodory property it would not matter if the triangles overlapped. It would be sufficient if we could reply *yes* to the following question (as also conjectured by Sebö [11]):

Unimodular Hilbert cover. *Is every discrete cone covered by its unimodular Hilbert subcones?*

Before our counterexamples there was no answer to this question in dimension > 3, and for dimensions 4 and 5 it remains unsolved. Moreover, a cone which possesses the discrete Carathéodory property, but which is not unimodular Hilbert covered, is also not yet known.

In order to delimit our problem it ought to be mentioned that each discrete cone possesses a unimodular triangulation, as long as no conditions, apart from finiteness, are imposed on the vectors that determine the triangulation.

THE COUNTEREXAMPLE

Even those experts who did not believe in the discrete Carathéodory property or the unimodular Hilbert cover were certainly surprised by the following counterexample which not only fulfills its purpose, but is also even aesthetically pleasing.

It is a 6-dimensional discrete cone which we denote by S_6. In order to make its symmetries more visible we select an embedding in 12-dimensional space. The row vectors $z_1, \dots, z_5$ and $z_6, \dots, z_{10}$ of the following matrices form the Hilbert basis of S_6:

$$\begin{bmatrix} 1\,0 & 1\,0\,0\,0\,0 & 0\,1\,2\,2\,1 \\ 1\,0 & 0\,1\,0\,0\,0 & 1\,0\,1\,2\,2 \\ 1\,0 & 0\,0\,1\,0\,0 & 2\,1\,0\,1\,2 \\ 1\,0 & 0\,0\,0\,1\,0 & 2\,2\,1\,0\,1 \\ 1\,0 & 0\,0\,0\,0\,1 & 1\,2\,2\,1\,0 \end{bmatrix} \qquad \begin{bmatrix} 0\,1 & 0\,2\,1\,1\,2 & 1\,0\,0\,0\,0 \\ 0\,1 & 2\,0\,2\,1\,1 & 0\,1\,0\,0\,0 \\ 0\,1 & 1\,2\,0\,2\,1 & 0\,0\,1\,0\,0 \\ 0\,1 & 1\,1\,2\,0\,2 & 0\,0\,0\,1\,0 \\ 0\,1 & 2\,1\,1\,2\,0 & 0\,0\,0\,0\,1 \end{bmatrix}$$

An embedding in 6-dimensional space is obtained, for example, when the columns $2, \dots, 7$ are selected. Each of the 12 coordinate hyperplanes of the 12-dimensional space intersects C_6 at one facet, since for every column the 5 vectors, whose components at that point are 0, are linearly independent. However, there are 15 further facets, 5 of which are not simplicial: they each contain 6 elements of $\mathrm{Hilb}(S_6)$.

The vectors $z_1, \dots, z_{10}$ lie in hyperplane H, which is defined by the equation $-6\,\zeta_1 - 6\,\zeta_2 + \zeta_3 + \cdots + \zeta_{12} = 1$. The cross-section of the convex cone C_6 generated by S_6 along H has the 5-dimensional volume $5/24$.

The discrete cone is not unimodular Hilbert covered, since the vector

$$t_1 = z_1 + \cdots + z_{10}$$

is not contained in any unimodular Hilbert subcone of S_6. Simultaneously, it is the "smallest" vector in S_6 with this property. If we take away the union of all unimodular Hilbert subcones from C_6, a convex cone surprisingly remains (its interior, to be more precise), which is generated by 22 vectors. Its cross-section along the hyperplane H has a 5-dimensional volume of $1/129600$; it therefore represents only $1/27000$ of the cross-section of C_6 with H.

However, t_1 does not disprove the discrete Carathéodory property. Due to the equation $z_1 + z_4 + z_7 + z_8 = 2z_5 + 2z_{10}$, t_1 can be written as an integer linear combination of 6 elements of the Hilbert basis:

$$t_1 = z_2 + z_3 + 3z_5 + z_6 + z_9 + 3z_{10}.$$

However, this is impossible for

$$t_2 = t_1 + z_2 + z_5 + 2\,(z_7 + \cdots + z_{10}).$$

(It can be shown that each element of S_6 can be represented by 7 elements of $\mathrm{Hilb}(S_6)$).

The obvious symmetry of S_6 can be described precisely without much effort. Both of the assignments given in the following diagrams can be realised by suitably permuting the coordinate axes of the 12-dimensional space. They therefore represent symmetries of S_6 (or C_6):

$$\sigma : z_1 \mapsto z_2 \mapsto z_3 \mapsto z_4 \mapsto z_5 \mapsto z_1, \quad z_6 \mapsto z_7 \mapsto z_8 \mapsto z_9 \mapsto z_{10} \mapsto z_6$$
$$\rho : z_1 \mapsto z_6 \mapsto z_1, \quad z_2 \mapsto z_8 \mapsto z_5 \mapsto z_9 \mapsto z_2, \quad z_3 \mapsto z_{10} \mapsto z_4 \mapsto z_7 \mapsto z_3.$$

(This is to be read as follows: σ transfers z_1 to z_2, z_2 to z_3, etc.). The group of motions determined by ρ and σ has 20 elements; other symmetries do not exist. σ and ρ^2 exchange the elements within the subsets $F_1 = \{z_1, \dots, z_5\}$ and $F_2 = \{z_6, \dots, z_{10}\}$, in the same way as in the symmetry group of the regular pentagon. On the other hand, ρ exchanges F_1 and F_2. Each element of $\mathrm{Hilb}(S_6)$ can therefore be transferred to any other so that S_6 looks "the same" at each boundary ray. The test vector t_1 is left invariant by σ and ρ (and its multiples are the only ones of this type in S_6); on the other hand, t_2 is only invariant for σ and ρ^2.

We have found a further counterexample beyond S_6. It is also 6-dimensional, but has a Hilbert basis with 12 vectors, and its symmetry group only has 4 elements. A counterexample of dimension $n + 1$ can easily be constructed from a counterexample of dimension n so that counterexamples are now also known in each dimension > 6. (An 0 is added to the vectors of the Hilbert basis as a new final component and then the vector $(0, \dots, 0, 1)$ is added to it).

Construction and tightening of discrete cones

In the search for cones without the discrete Carathéodory property or unimodular Hilbert covering one requires algorithms for three fundamental steps, i.e. for

1. the availability of candidates,
2. the test for unimodular Hilbert covering,
3. the test for the discrete Carathéodory property.

In this section we will discuss the first step, which is the most important of the three.

First of all the integer vectors $y_1, \dots, y_u$ are selected, whereby the number u and the components of y_i are either determined systematically or by a random number generator, whilst keeping to certain limits. The linear combinations $b_1 y_1 + \cdots + b_u y_u$ with coefficients $b_i \geq 0$ form a convex cone C, which can be described by inequations with integer coefficients. As desired, the integer vectors in C form a discrete cone S. Then the Hilbert basis of S is determined. We will refrain from describing how this is done. However, an efficient algorithm is absolutely necessary for the reduction procedure given below, as implemented by R. Koch and the author; see [6].

In general, the Hilbert basis of the discrete cones obtained is too large, rendering the second and third steps impossible, due to the inacceptable computing time and memory required—these calculations need to be repeated for numerous candidates. The reduction to *tight* cones was crucial for our success. This reduction will now be described.

In every boundary ray of a cone S there is exactly one element z of the Hilbert basis H (it is the first integer point on this ray if it is passed through from zero). We consider the cone C' which is spanned by the remaining elements $H' = \mathrm{Hilb}(S) \setminus \{z\}$. If H' is *not* the Hilbert basis of the discrete cone S', which is determined by C', we call z *destructive*. A cone should be called *tight* if every boundary element of the Hilbert basis is destructive. The tight cones are also precisely those cones that can not be reduced with the best possible preservation of the Hilbert basis.

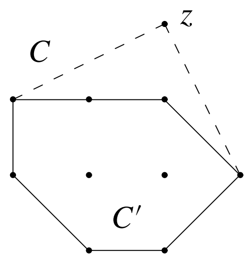

FIGURE 4. Tightening a cone

In order to "tighten" a cone the elements of the Hilbert basis are tested one at a time for destructivity. If a non-destructive element is found, it is simply discarded and work continues on the reduced cone.

A cone can often be completely "peeled off" using this procedure. If this is not the case, a tight cone remains which can then undergo further testing. We are not certain if a tight cone of dimension 3 exists. But the arguments of the previous section show that its Hilbert basis can not be contained in one plane. Tight cones can easily be found in dimension ≥ 4.

THE TEST FOR UNIMODULAR HILBERT COVERING

Our test for unimodular Hilbert covering is based on the dissection of the cone into elementary cells whose "walls" are formed by those hyperplanes that form the boundaries of the unimodular Hilbert subcones. Figure 5 shows this dissection as a cross-section of a three-dimensional cone. The following two alternatives are valid for each elementary cell E:

1. E is contained in a unimodular Hilbert subcone;
2. the interior of E is not intersected by any unimodular Hilbert subcone.

To test the unimodular Hilbert covering we first of all determine a Hilbert triangulation of candidate S, i.e. a triangulation in which every simplex is spanned by elements of the Hilbert basis. We can then immediately forget about the union of unimodular cones in this

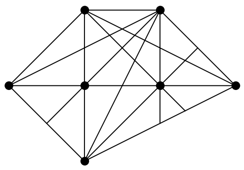

FIGURE 5. Dissection into elementary cells

triangulation—since it is already unimodular Hilbert covered. For each of its other cones D we then attempt to find a unimodular Hilbert subcone U which intersects the interior of D. If such a U can not be found, it is obvious that D is not unimodular covered. Otherwise one of the "walls" from U divides candidate D into two smaller cones, D_1 and D_2. Now the same procedure is applied recursively to D_1 and D_2. Since there are a finite number of unimodular Hilbert subcones, and therefore only a finite number of "walls", the case in which the tested candidate is contained in an elementary cell must occur at some point, so that one of the above alternatives apply.

This is (with some minor simplifications) our algorithm to test the unimodular Hilbert covering. Not only does it decide if S is unimodular covered, it also determines in the negative case all elementary cells that are not contained in any unimodular Hilbert subcone. (In the counterexample S_6 this is the case for exactly one elementary cell).

The discrete Carathéodory property can also be decided in a finite number of steps. However, this is not possible by means of convex geometry alone, and for this reason, we will refrain from describing the algorithm. Incidentally, we have not yet managed to realise this problem. It would only be worthwhile if we had an example that withstands all attacks with more basic weapons.

By trying out very many elements of S one could attempt to find one that can not be represented as a linear combination of $n = \dim(S)$ elements of the Hilbert basis. The search only needs to be undertaken in the elementary cells that are not unimodular covered, and this increases the chance of success considerably (with S_6 by factor 27000).

However, this method first of all led us to believe in the discrete Carathéodory property of S_6. This belief was then disproved by M. Henk, A. Martin and R. Weismantel (Magdeburg and Berlin) —the compiler we used had provided us with a poor random number generator. The results of the exact analysis of S_6, which was then carried out in cooperation, are described in [4].

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