Bounds on the Chvátal Rank of Polytopes in the 0/1-Cube

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Abstract. Gomory's and Chvátal's cutting-plane procedure proves recursively the validity of linear inequalities for the integer hull of a given polyhedron. The number of rounds needed to obtain all valid inequalities is known as the Chvátal rank of the polyhedron. It is well-known that the Chvátal rank can be arbitrarily large, even if the polyhedron is bounded, if it is of dimension 2, and if its integer hull is a 0/1-polytope. We prove that the Chvátal rank of polyhedra featured in common relaxations of many combinatorial optimization problems is rather small; in fact, the rank of any polytope contained in the n-dimensional 0/1-cube is at most $3n^2 \lg n$. This improves upon a recent result of Bockmayr et al. [6] who obtained an upper bound of $\mathrm{O}(n^3 \lg n)$.
Moreover, we refine this result by showing that the rank of any polytope in the 0/1-cube that is defined by inequalities with small coefficients is $\mathrm{O}(n)$. The latter observation explains why for most cutting planes derived in polyhedral studies of several popular combinatorial optimization problems only linear growth has been observed (see, e.g., [13]); the coefficients of the corresponding inequalities are usually small. Similar results were only known for monotone polyhedra before.
Finally, we provide a family of polytopes contained in the 0/1-cube the Chvátal rank of which is at least $(1+\epsilon)n$ for some $\epsilon > 0$; the best known lower bound was n.

1 Introduction

Chvátal [11] established cutting-plane proofs as a way to certify certain properties of combinatorial problems, e.g., to testify that there are no k pairwise non-adjacent nodes in a given graph, that there is no acyclic subdigraph with k arcs in a given digraph, or that there is no tour of length at most k in a prescribed instance of the traveling salesperson problem. In this paper we discuss the length of such proofs. Let us first recall the notion of a cutting-plane proof. A sequence of inequalities

$$c_1\, x \leqslant \delta_1, c_2\, x \leqslant \delta_2, \ldots , c_m\, x \leqslant \delta_m \tag{1}$$

is called a *cutting-plane proof* of $c\, x \leqslant \delta$ from a given system of linear inequalities $A\, x \leqslant b$, if $c_1, \ldots , c_m$ are integral, $c_m = c$, $\delta_m = \delta$, and if $c_i\, x \leqslant \delta_i'$ is a

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nonnegative linear combination of $A\,x \leqslant b, c_1\,x \leqslant \delta_1, \dots, c_{i-1}\,x \leqslant \delta_{i-1}$ for some δ_i' with $\lfloor \delta_i' \rfloor \leqslant \delta_i$. Obviously, if there is a cutting-plane proof of $c\,x \leqslant \delta$ from $A\,x \leqslant b$ then every integer solution to $A\,x \leqslant b$ must satisfy $c\,x \leqslant \delta$. Chvátal [11] showed that the converse holds as well. That is, if all integer points in a nonempty polytope $\{x \in \mathbb{R}^n : Ax \leqslant b\}$ satisfy an inequality $c\,x \leqslant \delta$, for some $c \in \mathbb{Z}^n$, then there is a cutting-plane proof of $c\,x \leqslant \delta$ from $A\,x \leqslant b$. Schrijver extended this result to rational polyhedra [36].

In a way, the sequential order of the inequalities in (1) obscures the (recursive) structure of the cutting-plane proof; it is better revealed by a directed graph with vertices $0, 1, 2, \dots, m$, in which an arc goes from node i to node j iff the i-th inequality has a positive coefficient in the linear combination of the j-th inequality. Here, 0 serves as a representative for any inequality in $A\,x \leqslant b$. The number of arcs in a longest simple path terminating at a node i is usually referred to as the *depth* of the i-th inequality $c_i\,x \leqslant \delta_i$ w.r.t. the cutting plane proof. The depth of the m-th inequality is called the *depth of the proof*, whereas m is the so-called *length* of the cutting-plane proof. We also say that an inequality $c\,x \leqslant \delta$ has depth (at most) d *relative to a polyhedron* $\{x : A\,x \leqslant b\}$ if it has a cutting-plane proof from $A\,x \leqslant b$ of depth less than or equal to d. The following theorem clarifies the relation between the depth and the length of a cutting-plane proof. It resembles very much the relation between the height and the number of nodes of a recursion tree where every interior node has at most degree n. It can be proved with the help of Farkas' Lemma.

Theorem 1 (Chvátal, Cook, and Hartmann [13]). *Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, let $A\,x \leqslant b$ have an integer solution, and let $c\,x \leqslant \delta$ have depth at most d relative to $A\,x \leqslant b$. Then there is a cutting-plane proof of $c\,x \leqslant \delta$ from $A\,x \leqslant b$ of length at most $(n^{d+1} - 1)/(n - 1)$.*

Gomory-Chvátal cutting-planes have gained importance for at least three reasons. First, the cutting-plane method is a (theoretical) tool to obtain a linear description of the integer hull of a polyhedron. In fact, as we already mentioned before any valid inequality for the integer hull has a cutting-plane proof from the defining system of the polyhedron. The *Chvátal rank* of this polyhedron is the smallest number d such that all inequalities valid for its integer hull have depth at most d relative to the defining system. Hence, if we later state lower and upper bounds for the depth of inequalities they immediately apply to the Chvátal rank of the corresponding polyhedron as well. Second, despite the early disappointments with Gomory's cutting-plane method [21, 22], it is of practical relevance. On the one hand, it has stimulated to a certain extent the search for problem-specific cutting planes which became the basis of an own branch of combinatorial optimization, namely polyhedral combinatorics (see, e.g., [33, 23, 35]). On the other hand, Balas et al. [2] successfully incorporated Gomory's mixed integer cuts within a Branch-and-Cut framework. Third, since cutting-plane theory implies that certain implications in integer linear programming have cutting-plane proofs, it is of particular importance in mathematical logic and complexity theory. It is a fundamental problem whether there exists a proof system for propositional logic in which every tautology has a short proof. Here, the length of the proof is measured by the total number of symbols in it and short means polynomial in the length of the tautology. This question is equivalent to whether or not NP equals co-NP. Cook, Coullard, and Turán [14] were the first to consider cutting-plane proofs as a propositional proof system. In particular, they pointed out that the cutting-plane proof system is a strengthening of resolution proofs. Since the work of Haken [25] exponential lower bounds are known for the latter. Results of Chvátal, Cook, and Hartmann [13], of Bonet, Pitassi, and Raz [7], of Impagliazzo, Pitassi, and Urquhart [30], and of Pudlák [34] imply exponential lower bounds on the length of cutting-plane proofs as well. On the other hand, there is no upper bound on the length of cutting-plane proofs in terms of the dimension of the corresponding polyhedron as the following well-known example shows. The Chvátal rank of the polytope defined by

$$\begin{aligned} -t\,x_1 + x_2 &\leqslant 1 \\ t\,x_1 + x_2 &\leqslant t + 1 \\ x_1 &\leqslant 1 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

grows with t. Here, t is an arbitrary positive number. This fact is rather counter-intuitive since the corresponding integer hull is a 0/1-polytope, i.e., all its vertices have components 0 or 1 only. That is, for any 0/1-polytope there is a simple certificate of the validity of an inequality $c\,x \leqslant \delta$. Just list all, at most 2^n possible assignments of 0/1-values to the variables. One of our main results helps to meet the natural expectation. We give a polynomial bound in the dimension for the Chvátal rank of any polytope contained in the 0/1-cube. Then, Theorem 1 implies the existence of exponentially long cutting-plane proofs, matching the known exponential lower bounds.

In polyhedral combinatorics, it has been quite common to consider the depth of a class of inequalities if not as an indicator of quality at least as a measure of its complexity. Hartmann, Queyranne, and Wang [29] give conditions under which an inequality has depth at most 1 and use them to establish that several classes of inequalities for the traveling salesperson polytopes have depth at least 2, as was claimed before in [3, 8, 9, 10, 18, 20, 24]. However, it follows from a recent result in [16] that deciding whether a given inequality $c\,x \leqslant \delta$ has depth at least 2 can in general not be done in polynomial time, unless P = NP. Chvátal, Cook, and Hartmann [13] (see also [27]) answered questions and proved conjectures of Schrijver, of Barahona, Grötschel, and Mahjoub [4], of Jünger, of Chvátal [12], and of Grötschel and Pulleyblank [24] on the behavior of the depth of certain inequalities relative to popular relaxations of the stable set polytope, the bipartite-subgraph polytope, the acyclic-subdigraph polytope, and the traveling salesperson polytope, resp. They obtained similar results for the set-covering and the set-partitioning polytope, the knapsack polytope, and the maximum-cut polytope, and so did Schulz [38] for the transitive packing, the clique partitioning, and the interval order polytope. The observed increase of the depth was never faster than a linear function of the dimension; we prove that this indeed has to be the case as the depth of any inequality with coefficients bounded by a constant is $\mathrm{O}(n)$, relative to a polytope in the 0/1-cube. Naturally, most polytopes associated with combinatorial optimization problems are 0/1-polytopes.

Main Results. We present two new upper bounds on the depth of inequalities relative to polytopes in the 0/1-cube. For notational convenience, let P be any polytope contained in the 0/1-cube, i.e., $P \subseteq [0,1]^n$, and let $c\,x \leqslant \delta$, $c \in \mathbb{Z}^n$ be an arbitrary inequality valid for the integer hull P_I of P.

We prove first that the depth of $c\,x \leqslant \delta$ relative to P is at most $2(n^2 + n \lg \|c\|_\infty)$. This yields an $\mathrm{O}(n^2 \lg n)$ bound on the Chvátal rank of P since any 0/1-polytope P_I can be represented by a system of inequalities $Ax \leqslant b$ with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ such that each absolute value of an entry in A is bounded by $n^{n/2}$. Note that the latter bound is sharp, i.e., there exist 0/1-polytopes with facets for which any inducing inequality $a\,x \leqslant \beta$, $a \in \mathbb{Z}^n$ satisfies $\|a\|_\infty \in \Omega(n^{n/2})$ [1].

Second, we show that the depth of $c\,x \leqslant \delta$ relative to P is no more than $\|c\|_1 + n$. A similar result was only known for monotone polyhedra [13]. In fact, we present a reduction to the monotone case that is of interest in its own right because of the smooth interplay of unimodular transformations and rounding operations. The second bound gives an asymptotic improvement by a factor n to the before-mentioned bound if the components of c are bounded by a constant.

Third, we construct a family of polytopes in the n-dimensional 0/1-cube whose Chvátal rank is at least $(1+\epsilon)n$, for some $\epsilon > 0$. In other words, if $r(n)$ denotes the maximum Chvátal rank over all polytopes that are contained in $[0,1]^n$, then it is one outcome of our study that this function behaves as follows:

$$(1+\epsilon)n \quad \leqslant \quad r(n) \quad \leqslant \quad 3n^2 \lg n \ .$$

Finally, we also show that the number of inequalities in any linear description of a polytope $P \subseteq [0,1]^n$ with empty integer hull is exponential in n, whenever there is an inequality of depth n.

Related Work. Via a geometric argument, Bockmayr and Eisenbrand [5] derived the first polynomial upper bound of $6\,n^3 \lg n$ on the Chvátal rank of polytopes in the n-dimensional 0/1-cube. Subsequently, Schulz [39] and Hartmann [28] independently obtained both a considerably simpler proof and a slightly better bound of $n^2 \lg(n^{n/2})$, by using bit-scaling. The reader is referred to the joint journal version of their papers [6], where the authors actually show that the depth of any inequality $c\,x \leqslant \delta$, $c \in \mathbb{Z}^n$, which is valid for P_I is at most $n^2 \lg \|c\|_\infty$, relative to P. For monotone polytopes P, Chvátal, Cook, and Hartmann [13] showed that the depth of any inequality $c\,x \leqslant \delta$ that is valid for P_I is at most $\|c\|_1$. Moreover, they also identified polytopes stemming from relaxations of combinatorial optimization problems that have Chvátal rank at least n.

Eventually, our study of $r(n)$ can also be seen as a continuation of the investigation of combinatorial properties of 0/1-polytopes, like their diameter [32], their number of facets [19], their number of vertices in a 2-dimensional projection [31], or their feature of admitting polynomial-time simplex-type algorithms for optimization [40].

The paper is organized as follows. We start with some preliminaries and introduce some notation in Section 2. We also show that any linear description of a polytope in the 0/1-cube that has empty integer hull and Chvátal rank n needs to contain at least 2^n inequalities. In Section 3, we prove the $\mathrm{O}(n^2 \lg n)$ upper bound on the Chvátal rank of polytopes in the 0/1-cube. Then, in Section 4, we utilize unimodular transformations as a key tool to derive an $\mathrm{O}(n)$ bound on the depth of inequalities with small coefficients, relative to polytopes in the 0/1-cube. Finally, we present the new lower bound on the Chvátal rank in Section 5.

2 Preliminaries

A *polyhedron* P is a set of points of the form $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$, for some matrix $A \in \mathbb{R}^{m\times n}$ and some vector $b \in \mathbb{R}^m$. The polyhedron is *rational* if both A and b can be chosen to be rational. If P is bounded, then P is called a *polytope*. The *integer hull* P_I of a polyhedron P is the convex hull of the integer points in P.

The *half space* $H = (c\,x \leqslant \delta)$ is the set $\{x \in \mathbb{R}^n \mid c\,x \leqslant \delta\}$, for some non-zero vector $c \in \mathbb{Q}^n$. It is called *valid* for a subset S of $\mathbb{R}^n$, if $S \subseteq H$. Sometimes we also say that the inequality $c\,x \leqslant \delta$ is valid for S. If the components of c are relatively prime integers, i.e., $c \in \mathbb{Z}^n$ and $\gcd(c) = 1$, then $H_I = (c\,x \leqslant \lfloor\delta\rfloor)$, where $\lfloor\delta\rfloor$ is the largest integer number less than or equal to δ. The *elementary closure* of a polyhedron P is the set

$$P' = \bigcap_{H \supseteq P} H_I,$$

where the intersection ranges over all rational half spaces containing P. We refer to an application of the $'$ operation as one iteration of the Gomory-Chvátal procedure. If we set $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geqslant 0$, then the Chvátal rank of P is the smallest number t such that $P^{(t)} = P_I$. The depth of an inequality $c\,x \leqslant \delta$ with respect to P is the smallest k such that $c\,x \leqslant \delta$ is valid for $P^{(k)}$.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron Q with $Q \supseteq P$ is called a *weakening* of P, if $Q_I = P_I$. If $c\,x \leqslant \delta$ is valid for P_I, then the depth of this inequality with respect to Q is an upper bound on the depth of this inequality with respect to P. It is easy to see that each polytope $P \subseteq [0,1]^n$ has a rational weakening in the 0/1-cube.

The following important lemma can be found in [37, p. 340]. (For a very nice treatment, see also [15, Lemma 6.33].) It allows to use induction on the dimension of the polyhedra considered and provides the key for the termination of the Gomory-Chvátal procedure, which was shown by Schrijver for rational polyhedra in [36].

Lemma 1. *Let F be a face of a rational polyhedron P. Then $F' = P' \cap F$.*

Lemma 1 yields the following upper bound on the Chvátal rank of rational polytopes in the 0/1-cube with empty integer hull (see [6] for details).

Lemma 2. *Let $P \subseteq [0,1]^n$ be a d-dimensional rational polytope in the 0/1-cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$; if $d > 0$, then $P^{(d)} = \emptyset$.*

Thus, if $cx \leqslant \delta$ is valid for a rational polytope $P \subseteq [0,1]^n$ and $cx \leqslant \delta - 1$ is valid for P_I, then $cx \leqslant \delta - 1$ is valid for $P^{(n)}$.

With these methods at hand one can prove the following result due to Hartmann [27].

Lemma 3. *If $P \subseteq [0,1]^n$ is a polytope and $\sum_{i \in I} x_i - \sum_{j \in J} x_j \leqslant r$ is valid for P_I for some subsets I and J of $\{1,\dots,n\}$, then this inequality has depth at most n^2 with respect to P.*

A side-product of our result in Section 4.3 is a reduction of this bound to $2n$.

Chvátal, Cook, and Hartmann [13, p. 481] provided the following family of rational polytopes in the 0/1-cube with empty integer hull and Chvátal rank n:

$$P_n = \Big\{x \in \mathbb{R}^n \mid \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geqslant \frac{1}{2}, \text{ for all } J \subseteq \{1,\dots,n\}\Big\}. \qquad (2)$$

The polytopes in this example have exponentially many inequalities, and this indeed has to be the case.

Proposition 1. *Let $P \subseteq [0,1]^n$ be a polytope in the 0/1-cube with $P_I = \emptyset$ and rank$(P) = n$. Any inequality description of P has at least 2^n inequalities.*

Proof. For a polytope $P \subseteq \mathbb{R}^n$ and for some $i \in \{1,\dots,n\}$ and $\ell \in \{0,1\}$ let $P_i^\ell \subseteq \mathbb{R}^{n-1}$ be the polytope defined by

$$P_i^\ell = \{x \in [0,1]^{n-1} \mid (x_1,\dots,x_{i-1},\ell,x_{i+1},\dots,x_n)^T \in P\}.$$

Notice that, if P is contained in a facet $(x_i = \ell)$ of $[0,1]^n$ for some $\ell \in \{0,1\}$ and some $i \in \{1,\dots,n\}$, then the Chvátal rank of P is the Chvátal rank of P_i^ℓ.

We will prove now that any one-dimensional face F_1 of the cube satisfies $F_1 \cap P \neq \emptyset$. We proceed by induction on n.

If $n = 1$, this is definitely true since P is not empty and since F_1 is the cube itself. For $n > 1$, observe that any one-dimensional face F_1 of the cube lies in a facet $(x_i = \ell)$ of the cube, for some $\ell \in \{0,1\}$ and for some $i \in \{1,\dots,n\}$. Since P has Chvátal rank n it follows that $\tilde{P} = (x_i = \ell) \cap P$ has Chvátal rank $n-1$. If the Chvátal rank of $\tilde{P}$ was less than that, P would vanish after $n-1$ steps. It follows by induction that $(F_1)_i^\ell \cap \tilde{P}_i^\ell \neq \emptyset$, thus $F_1 \cap P \neq \emptyset$.

Now, each 0/1-point has to be cut off from P by some inequality, as $P_I = \emptyset$. If an inequality $cx \leqslant \delta$ cuts off two different 0/1-points simultaneously, then it must also cut off a 1-dimensional face of $[0,1]^n$. Because of our previous observation this is not possible, and hence there is at least one inequality for each 0/1-point which cuts off only this point. Since there are 2^n different 0/1-points in the cube, the claim follows. □

We close this section by introducing some further notation. The ℓ_∞-*norm* $\|c\|_\infty$ of a vector $c \in \mathbb{R}^n$ is the largest absolute value of its entries, $\|c\|_\infty = \max\{|c_i| \mid i = 1, \dots, n\}$. The ℓ_1-*norm* $\|c\|_1$ of c is the sum $\|c\|_1 = \sum_{i=1}^n |c_i|$. We define the function $\lg : \mathbb{N} \to \mathbb{N}$ as

$$\lg n = \begin{cases} 1 & \text{if } n = 0 \\ 1 + \lfloor \log_2(n) \rfloor & \text{if } n > 0 \end{cases}$$

where $\lfloor y \rfloor$ denotes the largest integer smaller than or equal to y. Note that $\lg n$ is the number of bits in the binary representation of n. For a vector $x \in \mathbb{R}^n$, $\lfloor x \rfloor$ denotes the vector obtained by component-wise application of $\lfloor \cdot \rfloor$.

3 A New Upper Bound on the Chvátal Rank

We call a vector c *saturated* with respect to a polytope P, if $\max\{cx \mid x \in P\} = \max\{cx \mid x \in P_I\}$. If $Ax \leqslant b$ is an inequality description of P_I, then $P = P_I$ if and only if each row vector of A is saturated w.r.t. P. In [6], it is shown that an integral vector $c \in \mathbb{Z}^n$ is saturated after at most $n^2 \lg \|c\|_\infty$ steps of the Gomory-Chvátal procedure. Since each 0/1-polytope has a representation $Ax \leqslant b$ with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ such that each absolute value of an entry in A is bounded by $n^{n/2}$ (see, e.g., [33]), the known bound of $\mathrm{O}(n^3 \lg n)$ follows. One drawback in this proof is that faces of P which do not contain 0/1-points are taken to have worst case behavior n. The following observation is crucial to derive a better bound.

Lemma 4. *Let $cx \leqslant \alpha$ be valid for P_I and $cx \leqslant \gamma$ be valid for P, where $\alpha \leqslant \gamma$, $\alpha, \gamma \in \mathbb{Z}$ and $c \in \mathbb{Z}^n$. If, for each $\beta \in \mathbb{R}$, $\beta > \alpha$, the polytope $F_\beta = P \cap (cx = \beta)$ does not intersect with two opposite facets of the 0/1-cube, then the depth of $cx \leqslant \alpha$ is at most $2(\gamma - \alpha)$.*

Proof. Notice that $F'_\beta = \emptyset$ for each $\beta > \alpha$. The proof is by induction on $\gamma - \alpha$.

If $\alpha = \gamma$, there is nothing to prove. So let $\gamma - \alpha > 0$. Since $F'_\gamma = \emptyset$, Lemma 1 implies that $cx \leqslant \gamma - \epsilon$ is valid for P' for some $\epsilon > 0$ and thus the inequality $cx \leqslant \gamma - 1$ is valid for $P^{(2)}$. □

Proposition 2. *Let P be a rational polytope in the n-dimensional 0/1-cube. Any integral vector $c \in \mathbb{Z}^n$ is saturated w.r.t. $P^{(t)}$, for any $t \geqslant 2(n^2 + n \lg \|c\|_\infty)$.*

Proof. We can assume that $c \geqslant 0$ holds and that $P_I \neq \emptyset$. (It is shown in [6] that polytopes with empty integer hull have Chvátal rank at most n.) The proof is by induction on n and $\lg \|c\|_\infty$. The claim holds for $n = 1, 2$ since the Chvátal rank of a polytope in the 1- or 2-dimensional 0/1-cube is at most 4.

So let $n > 2$. If $\lg(\|c\|_\infty) = 1$, then the claim follows, e.g., from Theorem 3 below. So let $\lg \|c\|_\infty > 1$. Write $c = 2c_1 + c_2$, where $c_1 = \lfloor c/2 \rfloor$ and $c_2 \in \{0, 1\}^n$. By induction, it takes at most $2(n^2 + n \lg \|c_1\|_\infty) = 2(n^2 + n \lg \|c\|_\infty) - 2n$

iterations of the Gomory-Chvátal procedure until c_1 is saturated. Let $k = 2(n^2 + n \lg \|c\|_\infty) - 2n$.

Let $\alpha = \max\{c\,x \mid x \in P_I\}$ and $\gamma = \max\{c\,x \mid x \in P^{(k)}\}$. The integrality gap $\gamma - \alpha$ is at most n. This can be seen as follows. Choose $\hat{x} \in P^{(k)}$ with $c\,\hat{x} = \gamma$ and let $x_I \in P_I$ satisfy $c_1\,x_I = \max\{c_1\,x \mid x \in P^{(k)}\}$. One can choose x_I out of P_I since c_1 is saturated w.r.t. $P^{(k)}$. It follows that

$$\gamma - \alpha \leqslant c(\hat{x} - x_I) = 2c_1(\hat{x} - x_I) + c_2(\hat{x} - x_I) \leqslant n \ .$$

Consider now an arbitrary fixing of an arbitrary variable x_i to a specific value ℓ, $\ell \in \{0,1\}$. The result is the polytope

$$P_i^\ell = \{x \in [0,1]^{n-1} \mid (x_1, \dots, x_{i-1}, \ell, x_{i+1}, \dots, x_n)^T \in P\}$$

in the $(n-1)$-dimensional $0/1$-cube for which, by the induction hypothesis, the vector $\widetilde{c}_i = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)$ is saturated after at most

$$2((n-1)^2 + (n-1)\lg \|\widetilde{c}_i\|_\infty) \leqslant 2(n^2 + n \lg \|c\|_\infty) - 2n$$

iterations.

It follows that

$$\alpha - \ell c_i \geqslant \max\{\widetilde{c}_i\,x \mid x \in (P_i^\ell)^{(k)}\} = \max\{\widetilde{c}_i\,x \mid x \in (P_i^\ell)_I\}.$$

If $\beta > \alpha$, then $(c\,x = \beta) \cap P^{(k)}$ cannot intersect with a facet of the cube, since a point in $(c\,x = \beta) \cap P^{(k)} \cap (x_i = \ell)$, $\ell \in \{0,1\}$, has to satisfy $c\,x \leqslant \alpha$.

With Lemma 4, after $2n$ more iterations of the Gomory-Chvátal procedure, c is saturated, which altogether happens after $2(n^2 + n \lg \|c\|_\infty)$ iterations. □

We conclude this section with a new upper bound on the Chvátal rank.

Theorem 2. *The Chvátal rank of a polytope in the n-dimensional $0/1$-cube is $\mathrm{O}(n^2 \log n)$.*

Proof. Each polytope Q in the $0/1$-cube has a rational weakening P. The integral $0/1$-polytope P_I can be described by a system of integral inequalities $P_I = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ such that each absolute value of an entry in A is bounded by $n^{n/2}$. We estimate the number of Gomory-Chvátal steps until all row-vectors of A are saturated. Proposition 2 implies that those row-vectors are saturated after at most $2(n^2 + n \lg n^{n/2}) \leqslant 3\,n^2 \lg n$ steps. □

4 A Different Upper Bound on the Depth

In this section we show that any inequality $c\,x \leqslant \delta$, which is valid for the integer hull of a polytope P in the n-dimensional $0/1$-cube, has depth at most $n + \|c\|_1$ w.r.t. P.

We start by recalling some useful properties of monotone polyhedra, prove then that the Gomory-Chvátal operation is compliant with unimodular transformations, and eventually reduce the general case to the depth of inequalities over monotone polytopes via a special unimodular transformation.

4.1 Monotone Polyhedra

A nonempty polyhedron $P \subseteq \mathbb{R}^n_{\geqslant 0}$ is called *monotone* if $x \in P$ and $0 \leqslant y \leqslant x$ imply $y \in P$. Hammer, Johnson, and Peled [26] observed that a polyhedron P is monotone if and only if P can be described by a system $x \geqslant 0,\ Ax \leqslant b$ with $A, b \geqslant 0$.

The next statements are proved in [27] and [13, p. 494]. We include a proof of Lemma 6 for the sake of completeness.

Lemma 5. *If P is a monotone polyhedron, then P' is monotone as well.*

Lemma 6. *Let P be a monotone polytope in the 0/1-cube and let $w\,x \leqslant \delta$, $w \in \mathbb{Z}^n$, be valid for P_I. Then $w\,x \leqslant \delta$ has depth at most $\|w\|_1 - \delta$.*

Proof. The proof is by induction on $\|w\|_1$. If $\|w\|_1 = 0$, the claim follows trivially. W.l.o.g., we can assume that $w \geqslant 0$ holds. Let $\gamma = \max\{w\,x \mid x \in P\}$ and let $J = \{j \mid w_j > 0\}$. If $\max\{\sum_{j \in J} x_j \mid x \in P\} = |J|$, then, since P is monotone, $\hat{x}$ with

$$\hat{x}_i = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{otherwise} \end{cases}$$

is in P. Also $w\,\hat{x} = \gamma$ must hold. So $\gamma = \delta$ and the claim follows trivially. If $\max\{\sum_{j \in J} x_j \mid x \in P\} < |J|$, then $\sum_{j \in J} x_j \leqslant |J| - 1$ has depth at most 1. If $\|w\|_1 = 1$ this also implies the claim, so assume $\|w\|_1 \geqslant 2$. By induction the valid inequalities $w\,x - x_j \leqslant \delta,\ j \in J$ have depth at most $\|w\|_1 - \delta - 1$. Adding up the inequalities $w\,x - x_j \leqslant \delta,\ j \in J$ and $\sum_{j \in J} x_j \leqslant |J| - 1$ yields

$$w\,x \leqslant \delta + (|J| - 1)/|J|.$$

Rounding down yields $w\,x \leqslant \delta$ and the claim follows. □

4.2 Unimodular Transformations

Unimodular transformations and in particular switching operations will play a crucial role to relate the Chvátal rank of arbitrary polytopes in the 0/1-cube to the Chvátal rank of monotone polytopes. In this section, we show that unimodular transformations and the Gomory-Chvátal operation commute.

A *unimodular transformation* is a mapping

$$\begin{aligned} u : \mathbb{R}^n &\rightarrow \quad \mathbb{R}^n \\ x &\mapsto Ux + v, \end{aligned}$$

where $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix, i.e., $\det(U) = \pm 1$, and $v \in \mathbb{Z}^n$.

Note that u is a bijection. Its inverse is the unimodular transformation $u^{-1}(x) = U^{-1}x - U^{-1}v$. Since $U^{-1} \in \mathbb{Z}^{n \times n}$, u is also a bijection of $\mathbb{Z}^n$.

Consider the rational halfspace $(c\,x \leqslant \delta),\ c \in \mathbb{Z}^n, \delta \in \mathbb{Q}$. The set $u(c\,x \leqslant \delta)$ is the rational halfspace

$$\begin{aligned}\{x \in \mathbb{R}^n \mid c\,u^{-1}(x) \leqslant \delta\} &= \{x \in \mathbb{R}^n \mid c\,U^{-1}x \leqslant \delta + c\,U^{-1}v\} \\ &= (c\,U^{-1}x \leqslant \delta + c\,U^{-1}v).\end{aligned}$$

Notice that the vector $c\,U^{-1}$ is also integral. Let S be some subset of $\mathbb{R}^n$. It follows that $(c\,x \leqslant \delta) \supseteq S$ if and only if $(c\,U^{-1}x \leqslant \delta + c\,U^{-1}v) \supseteq u(S)$.

Consider now the first elementary closure P' of some polyhedron P,

$$P' = \bigcap_{\substack{(c\,x \leqslant \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (c\,x \leqslant \lfloor \delta \rfloor).$$

It follows that

$$u(P') = \bigcap_{\substack{(c\,x \leqslant \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (c\,U^{-1}x \leqslant \lfloor \delta \rfloor + c\,U^{-1}v).$$

From this one can derive the next lemma.

Lemma 7. *Let P be a polyhedron and u be a unimodular transformation. Then*

$$u(P') = (u(P))'.$$

Corollary 1. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $c\,x \leqslant \delta$ be a valid inequality for P_I. Let u be a unimodular transformation. The inequality $c\,x \leqslant \delta$ is valid for $P^{(k)}$ if and only if $u(c\,x \leqslant \delta)$ is valid for $(u(P))^{(k)}$.*

The *i-th switching operation* is the unimodular transformation

$$\begin{aligned}\pi_i : \quad \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_n),\end{aligned}$$

It has a representation

$$\begin{aligned}\pi_i : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto Ux + e_i,\end{aligned}$$

where U coincides with the identity matrix I_n except for $U_{(i,i)}$ which is -1. Note that the switching operation is a bijection of $[0,1]^n$. For the set $(c\,x \leqslant \delta)$ one has $\pi_i(c\,x \leqslant \delta) = \widetilde{c}\,x \leqslant \delta - c_i$. Here $\widetilde{c}$ coincides with c except for a change of sign in the i-th component.

4.3 The Reduction to Monotone Weakenings

If one wants to examine the depth of a particular inequality with respect to a polytope $P \subseteq [0,1]^n$, one can apply a series of switching operations until all its coefficients become nonnegative. An inequality with nonnegative coefficients defines a (fractional) 0/1-knapsack polytope K. The depth of this inequality with respect to the convex hull of P and K is then an upper bound on the depth with respect to P. We will show that $\text{conv}(P,K)^{(n)}$ has a monotone weakening in the 0/1-cube.

Lemma 8. *Let $P \subseteq [0,1]^n$ be a polytope in the 0/1-cube, with $P_I = K_I$, where $K = \{x \mid cx \leqslant \delta,\ 0 \leqslant x \leqslant 1\}$ and $c \geqslant 0$. Then, $P^{(n)}$ has a rational, monotone weakening Q in the 0/1-cube.*

Proof. We can assume that P is rational. Let $\hat{x}$ be a 0/1-point which is not contained in P, i.e., $c\hat{x} > \delta$. Let $I = \{i \mid \hat{x}_i = 1\}$. The inequality $\sum_{i \in I} x_i \leqslant |I|$ is valid for the cube and thus for P. Since $c \geqslant 0$, the corresponding face $F = \{x \mid \sum_{i \in I} x_i = |I|,\ x \in P\}$ of P does not contain any 0/1-points. Lemma 2 implies that $\sum_{i \in I} x_i \leqslant |I| - 1$ is valid for $P^{(n)}$.

Thus, for each 0/1-point $\hat{x}$ which is not in P, there exists a nonnegative rational inequality $a_{\hat{x}}\, x \leqslant \gamma_{\hat{x}}$ which is valid for $P^{(n)}$ and which cuts $\hat{x}$ off. Thus

$$\begin{aligned} &0 \leqslant x_i \leqslant 1,\ \ i \in \{1, \dots, n\} \\ &a_{\hat{x}}\, x \leqslant \gamma_{\hat{x}}, \quad \hat{x} \in \{0,1\}^n,\ \hat{x} \notin P \end{aligned}$$

is the desired weakening. □

Theorem 3. *Let $P \subseteq [0,1]^n$, $P \neq \emptyset$ be a nonempty polytope in the 0/1-cube and let $cx \leqslant \delta$ be a valid inequality for P_I with $c \in \mathbb{Z}^n$. Then $cx \leqslant \delta$ has depth at most $n + \|c\|_1$ with respect to P.*

Proof. One can assume that c is nonnegative, since one can apply a series of switching operations. Notice that this can change the right hand side δ, but in the end δ has to be nonnegative since $P \neq \emptyset$. Let $K = \{x \in [0,1]^n \mid cx \leqslant \delta\}$ and consider the polytope $Q = \text{conv}(K,P)$. The inequality $cx \leqslant \delta$ is valid for Q_I and the depth of $cx \leqslant \delta$ with respect to P is at most the depth of $cx \leqslant \delta$ with respect to Q. By Lemma 8, $Q^{(n)}$ has a monotone weakening S. The depth of $cx \leqslant \delta$ with respect to $Q^{(n)}$ is at most the depth of $cx \leqslant \delta$ with respect to S. But it follows from Lemma 6 that the depth of $cx \leqslant \delta$ with respect to S is at most $\|c\|_1 - \delta \leqslant \|c\|_1$. □

5 A New Lower Bound on the Chvátal Rank

To the best of the authors' knowledge, no example of a polytope P in the n-dimensional 0/1-cube with $\text{rank}(P) > n$ has been provided in the literature so far. We now show that $r(n) > (1 + \epsilon)n$, for infinitely many n, where $\epsilon > 0$.

The construction relies on the lower bound result for the fractional stable-set polytope due to Chvátal, Cook, and Hartmann [13].

Let $G = (V, E)$ be a graph on n vertices, $\mathcal{C}$ be the family of all cliques of G, and let $Q \subseteq \mathbb{R}^n$ be the fractional stable set polytope of G defined by the equations

$$\begin{aligned} x(C) &\leqslant 1 \text{ for all } C \in \mathcal{C}, \\ x_v &\geqslant 0 \text{ for all } v \in V. \end{aligned} \tag{3}$$

Let $\mathbf{e}$ be the vector of all ones. The following lemma is proved in [13, Proof of Lemma 3.1].

Lemma 9. *Let $k < s$ be positive integers and let G be a graph with n vertices such that every subgraph of G with s vertices is k-colorable. If P is a polyhedron that contains Q_I and the point $u = \frac{1}{k}\mathbf{e}$, then $P^{(j)}$ contains the point $x^j = (\frac{s}{s+k})^j u$.*

Let $\alpha(G)$ be the size of the largest independent subset of the nodes of G. It follows that $\mathbf{e}\,x \leqslant \alpha(G)$ is valid for Q_I. One has

$$\mathbf{e}\,x^j = \frac{n}{k}(\frac{s}{s+k})^j \geqslant \frac{n}{k}e^{-jk/s} \quad ,$$

and thus x^j does not satisfy the inequality $\mathbf{e}\,x \leqslant \alpha(G)$ for all $j < (s/k)\ln\frac{n}{k\alpha(G)}$.

Erdös proved in [17] that for every positive t there exist a positive integer c, a positive number δ and arbitrarily large graphs G with n vertices, cn edges, $\alpha(G) < tn$ and every subgraph of G with at most δn vertices is 3 colorable. One wants that $\ln\frac{n}{k\alpha(G)} \geqslant 1$ and that s/k grows linearly, so by chosing some $t < 1/(3e)$, $k = 3$ and $s = \lfloor \delta n \rfloor$ one has that x^j does not satisfy the inequality $\mathbf{e}\,x \leqslant \alpha(G)$ for all $j < (s/k)$.

We now give the construction. Let P be the polytope that results from the convex hull of P_n defined in (2) and Q. $P_n \subseteq P$ contributes to the fact that $1/2\,\mathbf{e}$ is in $P^{(n-1)}$ [13, Lemma 7.2]. Thus $x_0 = 1/3\,\mathbf{e}$ is in $P^{(n-1)}$. Since the convex hull of P is Q_I, it follows from the above discussion that the depth of $\mathbf{e}\,x \leqslant \alpha(G)$ with respect to $P^{(n-1)}$ is $\Omega(n)$. Thus the depth of $\mathbf{e}\,x \leqslant \alpha(G)$ is at least $(n-1) + \Omega(n) \geqslant (1+\epsilon)n$ for infinitely many n, where $\epsilon > 0$.

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