ACYCLIC FLOW POLYTOPES AND KOSTANT'S PARTITION FUNCTION

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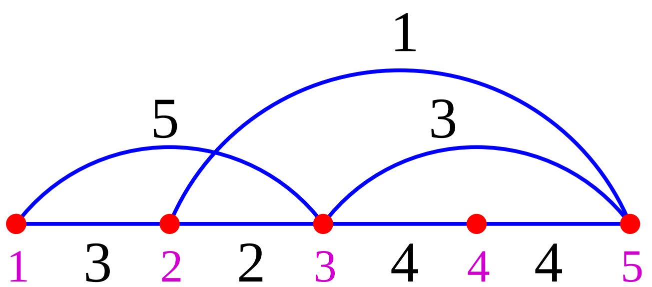
(with A. Postnikov)



N- flow
size = 8
V(G) = *{1, ..., m+1}*

flow polytope $\mathcal{F}_G \subset \mathbb{R}^{E(G)}$:

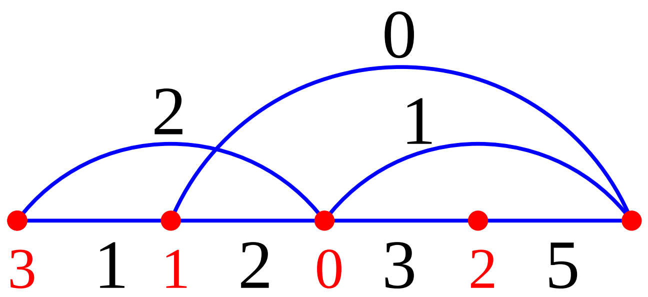
flows $f : E(G) \to \mathbb{R}_{\geq 0}$ of size 1

vertices $\leftrightarrow$ **paths** in G from 1 to $m+1$

12345 1235 125 1345 135

excess flow vector

$$\gamma = (a_1, \ldots, a_m) \in \mathbb{N}^m$$



γ = (3, 1, 0, 2)

(restricted) Kostant's partition function:

$$\boldsymbol{e_i} = (0 \cdots 0 \overset{i}{1} 0 \cdots 0) \in \mathbb{R}^{m+1}$$

$$\boldsymbol{e_{ij}} = e_i - e_j$$

$$\boldsymbol{\nu} \in \mathbb{Z}^{m+1}, \quad \sum \nu_i = 0$$

$$\boldsymbol{A_m^+} = \{e_{ij} : 1 \leq i < j \leq m+1\} \subset \mathbb{Z}^{m+1}$$

$$\boldsymbol{S} \subseteq A_m^+$$

$$\boldsymbol{K_S(\nu)} = \# \left\{ \left(b_{ij}\right)_{e_{ij} \in S} : \nu = \sum b_{ij} e_{ij} \right\}$$

Example. $S = \{e_{12}, e_{23}, e_{13}\}$

$$(2,0,-2) = 2e_{12}+2e_{23} = e_{12}+e_{13}+e_{23} = 2e_{13}$$

$$\Rightarrow K(2,0,-2) = 3.$$

Proposition. *Let*

$$S = S(G) = \{e_{ij} \,:\, (i,j) \in E(G)\}.$$

The number of $\mathbb{N}$-flows with excess flow $(a_1, \ldots, a_m)$ is equal to

$$K_S\left(a_1, \ldots, a_m, -\sum a_i\right).$$

Main Theorem (D. Peterson, unpublished, for $S = A_m^+$). *Let*

$$\boldsymbol{d_i} = \mathrm{outdeg}(i) - 1.$$

Then

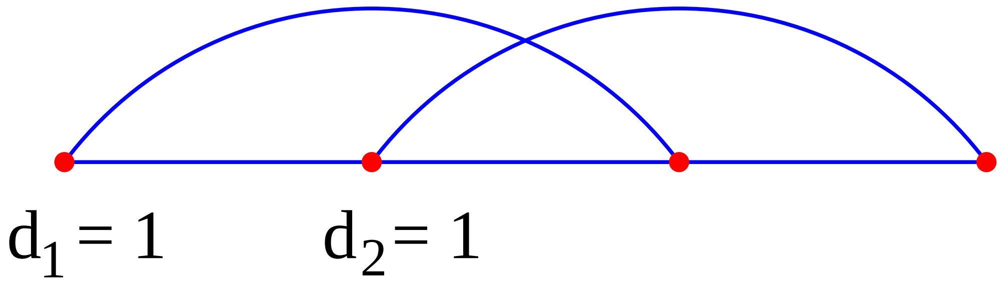
$$K_S\left(a_1, \dots, a_m, -\sum a_i\right) =$$

$$\sum K_S(\nu_1 - d_1, \dots, \nu_{m-1} - d_{m-1}) \cdot \binom{a_1 + d_1}{\nu_1} \cdots \binom{a_{m-1} + d_{m-1}}{\nu_{m-1}},$$

summed over all $\nu_1, \dots, \nu_{m-1} \in \mathbb{N}$ *satisfying*

$$\nu_1 + \cdots + \nu_i \geq d_1 + \cdots + d_i$$

$$\sum \nu_i = d_1 + \cdots + d_{m-1}.$$



$$(\nu_1, \nu_2) = (2,0), (1,1)$$
$$S = \{e_{12}, e_{13}, e_{23}, e_{24}, e_{34}\}$$

$$K_S(a,b,c,-a-b-c) =$$
$$K_S(1,-1)\binom{a+1}{2} + K_S(0,0)\binom{a+1}{1}\binom{b+1}{1}$$
$$= \binom{a+1}{2} + (a+1)(b+1).$$

Idea of proof. Consider

$$\prod_{(i,j)\in E(G)} \frac{1}{1-x_i x_j^{-1}} =$$

$$\sum_{\beta} K_S(\beta) x^{\beta}.$$

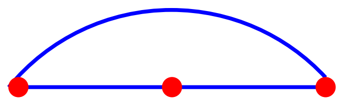
Systematically apply the identity

$$\frac{1}{1-x_i x_j^{-1}} \cdot \frac{1}{1-x_j x_k^{-1}} =$$

$$\frac{1}{1-x_i x_k^{-1}} \left(\frac{x_i x_j^{-1}}{1-x_i x_j^{-1}} + \frac{1}{1-x_j x_k^{-1}} \right)$$

(Elliott-MacMahon algorithm).

Example.



Good way: $$\frac{1}{(1-xy^{-1})(1-yz^{-1})(1-xz^{-1})}$$

$$= \frac{1}{(1-xz^{-1})^2}\left(\frac{xy^{-1}}{1-xy^{-1}} + \frac{1}{1-yz^{-1}}\right)$$

$$= \text{bad terms} + \frac{1}{(1-xz^{-1})^2(1-yz^{-1})}$$

$$= \text{bad terms} + \sum_{a,b\geq 0} (a+1)x^a y^b z^{-a-b}$$

$$\Rightarrow K(a,b,-a-b) = a+1.$$

Bad way: $$\frac{1}{(1-xy^{-1})(1-yz^{-1})(1-xz^{-1})}$$

$$= \frac{1}{(1-xz^{-1})^2}\left(\frac{1}{1-xy^{-1}} + \frac{yz^{-1}}{1-yz^{-1}}\right)$$

Corollary. *Let $d = \dim \mathcal{F}_G$. Then*

$$d! \cdot \mathrm{vol}(\mathcal{F}_G) := \tilde{V}(\mathcal{F}_G)$$

$$= K_S\left(d_{m-1}, d_{m-2}, \ldots, d_1, -\sum d_i\right).$$

For $G = K_{m+1}$, we have

$$\tilde{V}(\mathcal{F}_{K_{m+1}}) = K\left(1, 2, \ldots, m-2, -\binom{m-1}{2}\right).$$

Chan-Robbins conjecture:

Theorem (Zeilberger). *We have*

$$\tilde{V}\left(\mathcal{F}_{K_{m+1}}\right) = C_1 \cdots C_{m-2},$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$ **(Catalan number)**.

Note. The polytope $\mathcal{F}_{K_{m+1}}$ is "unimodularly equivalent" to the polytope of all $m \times m$ doubly-stochastic matrices A satisfying $A_{ij} = 0$ if $j > i + 1$.

$$\begin{bmatrix} .3 & .7 & & \\ .2 & .1 & .7 & \\ .4 & 0 & .2 & .4 \\ .1 & .2 & .1 & .6 \end{bmatrix}$$

Alternate form of Chan-Robbins conjecture: Let $f(n)$ be the number of $n \times n$ $\mathbb{N}$-matrices A such that

- $A_{ij} = 0$ if $j > i + 1$
- row and column sum vector

$$\left(1, 3, 6, \ldots, \binom{n+1}{2}\right)$$

$$\begin{array}{lllll} 0 & 1 & & & \\ 0 & 1 & 2 & & \\ 0 & 0 & 2 & 4 & \\ 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 3 & 10 \end{array}$$

Then $f(n) = C_1 C_2 \cdots C_n$.

Can also do

$$K\left(a, a+1, \ldots, b, -\binom{b+1}{2} + \binom{a}{2}\right).$$

E.g.,

$$K(a, a+1, a+2, a+3, -4a-6) =$$

$$\frac{(a+1)(2a+3)(a+2)^2(2a+5)(a+3)}{72}$$

$$= C_1C_2C_3 \cdot [\text{\# plane partitions}$$

$$\text{of shape} \subseteq (3,2,1), \text{largest part} \leq a]$$

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 0 \\ \hline 3 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} 1 & \multicolumn{1}{|c}{} & \\ \cline{1-1}\end{array}$$

E-M algorithm induces a triangulation of $\mathcal{F}_{G^+}$ and a "nice" decomposition of $(\mathbb{N},\gamma)$-flows in G.

E.g.,

$$\sum_{1\leq i<j\leq 4} \gamma_{ij} e_{ij}$$

is one of:

$$\mathbb{N}e_{124}+\mathbb{N}e_{134}+\mathbb{N}e_{234}+\mathbb{N}e_{14}+\mathbb{N}e_{24}+\mathbb{N}e_{34}$$

or

$$e_{12}+\mathbb{N}e_{1234}+\mathbb{N}e_{134}+\mathbb{N}e_{234}+\mathbb{N}e_{14}+\mathbb{N}e_{234}+\mathbb{N}e_{34},$$

where

$$\mathbf{e_{i_1 i_2 \cdots i_k}} = e_{i_1 i_2} + e_{i_2 i_3} + \cdots + e_{i_{k-1} i_k}.$$

Alternate form of volume computation:

Start with

$$\mathbf{M} = \prod_{(i,j)\in E(G)} x_{ij}.$$

Apply the rule

$$x_{ij}x_{jk} \to x_{ik}\left(x_{ij} + x_{jk}\right)$$

until unable to do so, obtaining a polynomial $P(x_1, \ldots, x_{m+1})$. Then

$$P(1, 1, \ldots, 1) = \tilde{V}\left(\mathcal{F}_{G^{++}}\right),$$

where $\mathbf{G}^{++}$ is G with vertices 0 and m+2 adjoined, and new edges $(0, i)$ and $(i, m+1)$.

Example. $M = x_{12}x_{23}\cdots x_{m,m+1}$

$\Rightarrow P(1,1,\ldots,1) = C_m$ (Catalan number)

$$M = \prod_{1\le i<j\le m+1} x_{ij}$$

$$\Rightarrow P(1,1,\ldots,1) = C_1C_2\cdots C_m.$$

Theorem (easy). *$K\left(a_1, \ldots, a_m, -\sum a_i\right)$ is divisible by*

$$(a_1+1)(a_1+2)\cdots(a_1+m-1).$$

Theorem (J. R. Schmidt and A. M. Bincer, 1984; A. N. Kirillov, 1999) *Also divisible by*

$$a_1+a_2+\cdots+a_{m-2}+3a_{m-1}+3.$$

In fact,

$$3K\left(a_1, \ldots, a_m, -\sum a_i\right) = (a_1+\cdots+a_{m-2}+3a_{m-1}+3) \cdot K_{\mathbf{no\ e_{m-1,m}}}\left(a_1, \ldots, a_m, -\sum a_i\right).$$

Example.

$$K(a,b,c,d,-a-b-c-d) =$$

$$\frac{1}{360}(a+1)(a+2)(a+3)(a+b+3c+3)$$

$$\cdot(a^2+5ab+10b^2+9a+30b+20)$$

Open: Are all coefficients of

$$K_S\left(a_1,\ldots,a_m,-\sum a_i\right)$$

nonnegative?

EHRHART POLYNOMIALS

Let $\mathcal{P}$ be an integer polytope in $\mathbb{R}^n$ and let $r \geq 1$. Define

$$\mathbf{r}\mathcal{P} = \{rv \,:\, v \in \mathcal{P}\}$$

$$\mathbf{i}(\mathcal{P}, \mathbf{r}) = \#\left(r\mathcal{P} \cap \mathbb{Z}^n\right),$$

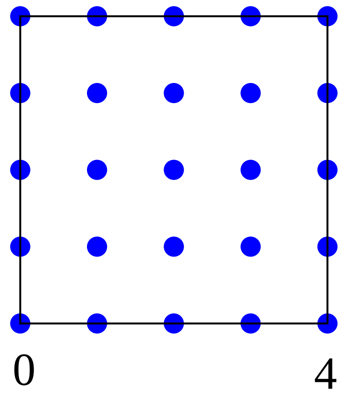
the **Ehrhart polynomial** of $\mathcal{P}$.

- $i(\mathcal{P}, r)$ is a polynomial in r
- $i(\mathcal{P}, 0) = 1$
- If $r > 0$, then

  $i(\mathcal{P}, -r) = (-1)^{\dim \mathcal{P}} \#(\mathrm{int}(r\mathcal{P}) \cap \mathbb{Z}^n)$
- $i(\mathcal{P}, r) = V(\mathcal{P}) r^n + O(r^{n-1})$.

Example. $\mathcal{P}$ = unit square:

$$0 \le x \le 1, \quad 0 \le y \le 1$$



$4\mathcal{P}$

$$\begin{aligned} \widetilde{V}(\mathcal{P}) &= 2! \cdot 1 \;=\; 2 \\ i(\mathcal{P}, r) &= (r+1)^2 \\ i(\mathcal{P}, -r) &= (r-1)^2 \end{aligned}$$

For $\mathcal{P} = \mathcal{F}_G$ and

$$S = \{e_{ij} \, : \, (i, j) \in E(G)\},$$

we have:

$$i(\mathcal{F}_G, r) = K_S(r, \underbrace{0, \ldots, 0}_{m-1}, -r)$$

MIXED VOLUMES

Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be convex polytopes (or any convex bodies) in $\mathbb{R}^n$. Let

$$t_1, \ldots, t_k \in \mathbb{R}_{\geq 0}.$$

Define the **Minkowski sum**

$$t_1\mathcal{P}_1+\cdots+t_k\mathcal{P}_k = \{t_1v_1+\cdots+t_kv_k : v_i \in \mathcal{P}_i\}.$$

Theorem (Minkowski) *There exist* $V(\mathcal{P}_1^{a_1}, \ldots, \mathcal{P}_k^{a_k}) \geq 0$ **(mixed volumes)** *such that*

$$V(t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k) = \sum_{a_1+\cdots+a_k=n} \binom{n}{a_1, \ldots, a_k} \cdot V(\mathcal{P}_1^{a_1}, \ldots, \mathcal{P}_k^{a_k}) t_1^{a_1} \cdots t_k^{a_k}.$$

MIXED LATTICE POINT ENUMERATORS

$\boldsymbol{\mathcal{P}_1, \ldots, \mathcal{P}_m}$ integer polytopes in $\mathbb{R}^d$

$$a_1, \ldots, a_m \in \mathbb{N}$$

$$\begin{aligned} \boldsymbol{\mathcal{P}} &= a_1\mathcal{P}_1 + \cdots + a_m\mathcal{P}_m \\ &= \{a_1v_1 + \cdots + a_mv_m \,:\, v_i \in \mathcal{P}_i\}. \end{aligned}$$

McMullen: Let

$$\boldsymbol{N(\mathcal{P})} = \#\left(\mathbb{Z}^d \cap \mathcal{P}\right),$$

the **mixed lattice point enumerator** of $\mathcal{P}_1, \ldots, \mathcal{P}_m$ ($m = 1$: Ehrhart polynomial). Then

$$N(\mathcal{P}) \in \mathbb{Q}[a_1, \ldots, a_m].$$

Let

$\boldsymbol{N(\mathcal{P}_d)}$ = terms of total degree d.

Then

$$\begin{aligned} N(\mathcal{P})_d &= \mathrm{vol}(\mathcal{P}) \\ &= \mathrm{vol}(a_1\mathcal{P}_1 + \cdots + a_m\mathcal{P}_m) \\ &= \sum_{i_1+\cdots+i_m=d} \binom{d}{i_1,\ldots,i_m} \\ &\qquad \cdot \underbrace{V(\mathcal{P}_1^{i_1},\ldots,\mathcal{P}_m^{i_m})}_{\textbf{mixed volume}}. \end{aligned}$$

Now let $V(G) = \{1, 2, \ldots, m+1\}$ and $\gamma = (a_1, \ldots, a_m)$ as before. Let

$$\mathbf{G_i} = G|_{i,i+1,\ldots,m+1}$$

$$\boldsymbol{\mathcal{F}_i} = \mathcal{F}_{G_i} \subset \mathbb{R}^{E(G)}.$$

Then

$$N(a_1\mathcal{F}_1 + \cdots + a_m\mathcal{F}_m) =$$

$$\#(\mathbb{N}, \gamma)\text{-flows in } G$$

$$= K_S\left(a_1, \ldots, a_m, -\sum a_i\right).$$

In particular,

$$V(a_1\mathcal{F}_1 + \cdots + a_m\mathcal{F}_m) =$$

$$\sum_{\text{certain } \nu_i \in \mathbb{N}} \underbrace{K_S(\nu_1 - d_1, \ldots, \nu_{m-1} - d_{m-1})}_{\textbf{mixed volumes}}$$

$$\cdot \frac{a_1^{\nu_1}}{\nu_1!} \cdots \frac{a_{m-1}^{\nu_{m-1}}}{\nu_{m-1}!}.$$