MATH 16C: MULTIVARIATE CALCULUS

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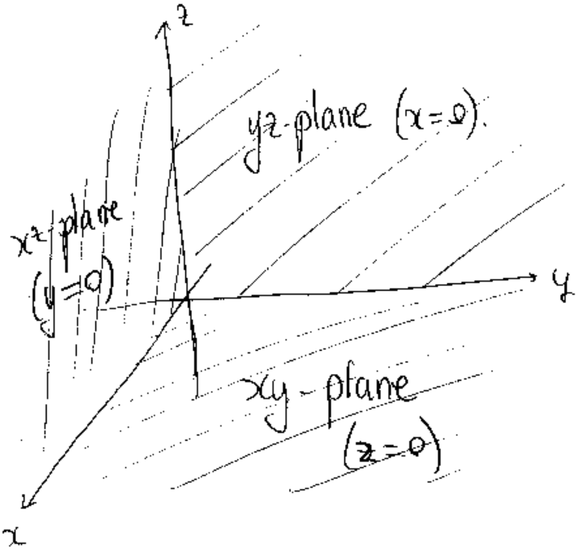
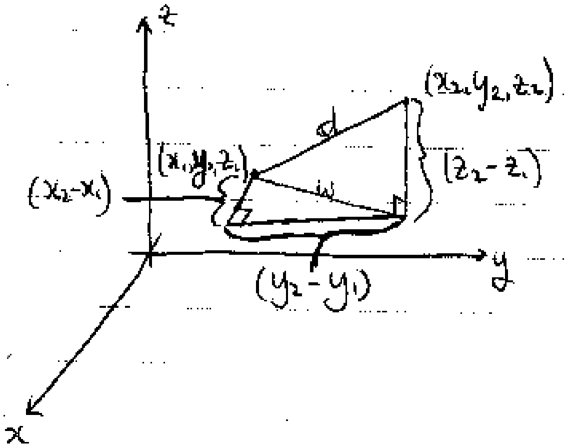
7.1: The 3-dimensional coordinate system





Thus, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. **EXAMPLE:** Find the distance between $(-3, 2, 5)$ and $(4, -1, 2)$.

$$d = \sqrt{((-3) - 4)^2 + (2 - (-1))^2 + (5 - 2)^2} = \sqrt{49 + 9 + 9} = \sqrt{67}.$$

Mid-point between two points

- What is the mid-point between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)?
- Let (x_m, y_m, z_m) be the mid-point. Then x_m must be half-way bewteen x_1 and x_2, so $x_m = \frac{x_1+x_1}{2}$.
- Also, y_m must be half-way between y_1 and y_2, so $y_m = \frac{y_1+y_2}{2}$
- Similarly, z_m must be half-way between z_1 and z_2, so $z_m = \frac{z_1+z_2}{2}$.
- Thus,

$$(x_m, y_m, z_m) = \left(\frac{x_1 + x_1}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

- E.g. What is the midpoint bewteen $(-3, 2, 5)$ and $(4, 1, 2)$?

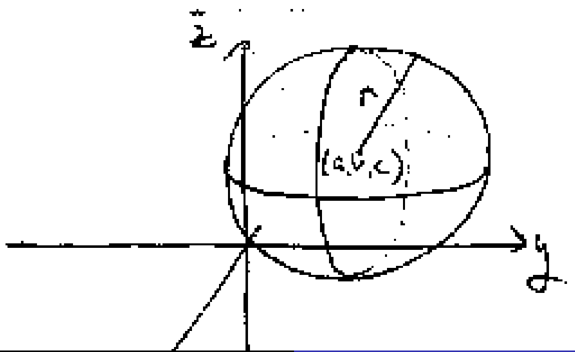
$$(x_m, y_m, z_m) = \left(\frac{(-3) + 4}{2}, \frac{2 + 1}{2}, \frac{5 + 2}{2} \right) = \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right).$$

Question:

If we fix a point (a, b, c), what is the set of all points at a distance r from (a, b, c) called?

Answer: A SPHERE centered at (a, b, c).

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = r \Leftrightarrow (x-a)^2+(y-b)^2+(z-c)^2 = r^2.$$



QUIZ: What is the equation of the sphere with center $(2, -3, 8)$ and radius 5?

$(A)(x-2)^2+(y+3)^2+(z-8)^2 = 25$ or $(B)(x+2)^2+(x-3)^2+(x+8)^2 = 5$

PROBLEM: Let $x^2 - 2x + y^2 + 4y + z^2 - 8z - 15 = 0$, find the center and radius of this sphere.

Rewrite the equation by **completing the square**:

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x - 1)^2 - 1$$
$$y^2 + 4y = (y^2 + 4y + 4) - 4 = (y + 2)^2 - 4$$
$$z^2 - 8z = (z^2 - 8z + 16) - 16 = (z - 4)^2 - 16$$

So,

$$x^2 - 2x + y^2 + 4y + z^2 - 8z - 15 = (x - 1)^2 - 1 + (y + 2)^2 - 4 + (z -$$
$$= (x - 1)^2 + (y + 2)^2 + (z - 4)^2 - 36$$

The equation of the sphere is $(x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 36$.

Theorem:

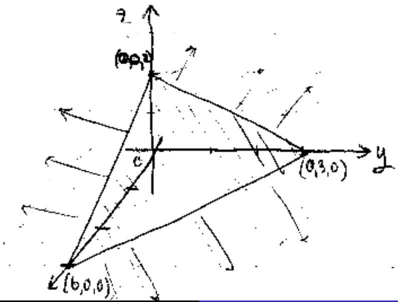
The equation of a plane in 3-dimensions is $Ax + By + Cz = D$ where A, B, C, D are constants.

How do you draw a plane?

We use x,y,z-axis intercepts to find 3 points on the plane.

EXAMPLE: Draw $x + 2y + 3z = 6$

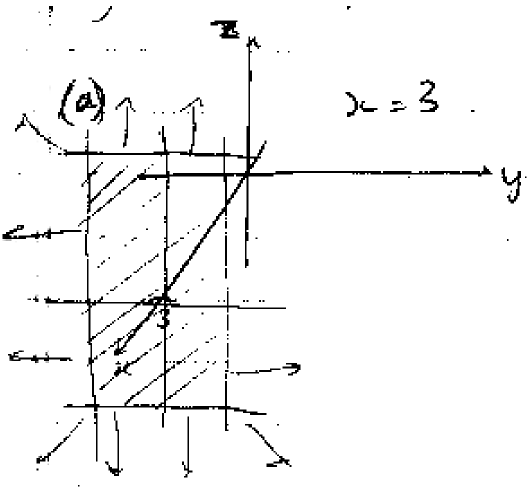
- x-intercept is $y = 0, z = 0 \Rightarrow x = 6$.
- y-intercept is $x = 0, z = 0 \Rightarrow y = 3$.
- z-intercept is $y = 0, x = 0 \Rightarrow z = 2$.



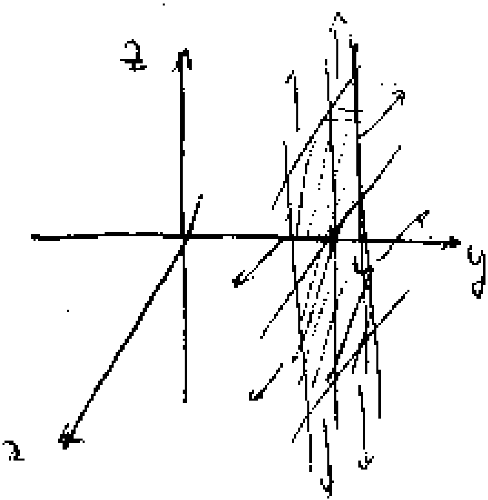
But, there aren't always x-axis or y-axis or z-axis intercepts!!

EXAMPLE: Draw the planes (a) $x = 3$, (b) $y = 4$.

(a) $x = 3$. The plane is parallel to the yz-plane. There are no y-axis or z-axis intercepts.



(b) $y = 4$. The plane is parallel to the xz-plane. There are no x-axis or z-axis intercepts.



Section 7.2: Surfaces

LAST EPISODE...

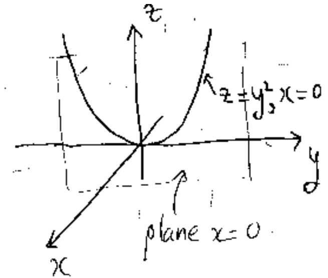
- **Theorem** Every quadric surface has an equation of the form:

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0$$

where A, B, C, D, E, F, G are constants.

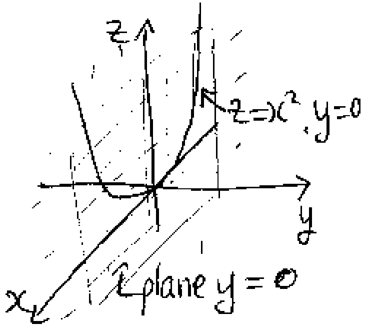
- CHALLENGE: Recognize the shape of the surface from Equation!!!!
- **Strategy:** Find and graph the intersection of the surface with the planes $x = c$, $y = c$, or $z = c$ where c is some constant. We SLICE the surface!! We do a CT-scan!!!
- The intersection of a surface with a plane $x = c$, $y = c$, or $z = c$ is called a **TRACE**.
- **Idea:** As we sweep the planes of intersections we get an idea of volume, we then join all slices to reconstruct a 3-D figure.
- **EXAMPLE:** Graph the quadric surface $z = x^2 + y^2$.

The intersection of the surface with the plane $x = 0$ is the parabola $z = y^2$ lying on the yz-plane. Graph it



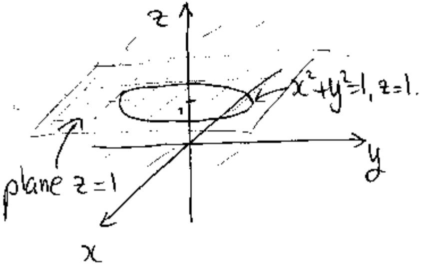
What happens if we slice with x=1? Result is $z = 1 + y^2$, another parabola.

Next Graph the intersection of the surface with the plane $y = 0$ is the parabola $z = x^2$ lying on the xz-plane.

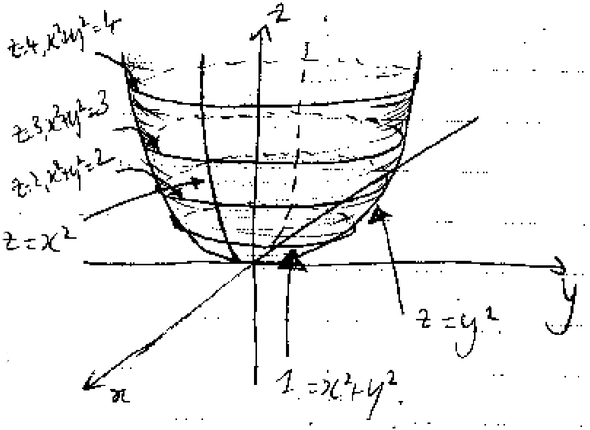


What happens if we slice with y=2?, another parabolar $z = x^2 + 4$.

Graph the intersection of the surface with the plane $z = 0$ is the point $(x, y, z) = (0, 0, 0)$ since $z = 0$ implies $0 = x^2 + y^2$. SUPER EASY, a point is Trace!! Graph the intersection of the surface with the plane $z = 1$ is the circle or radius one centered at $(0, 0, 1)$, which lies on the plane $z = 1$.

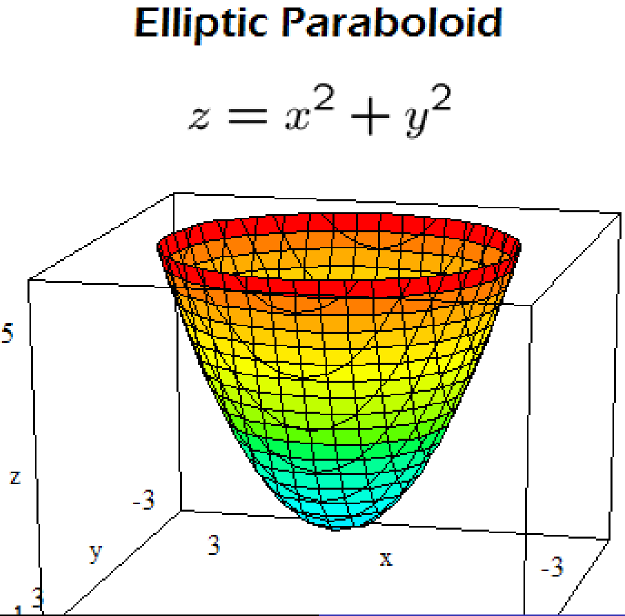


Similarly, graph the intersection of the surface with the planes $z = 2, 3, 4, \ldots$ each one gives a circle of radius $\sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$ with center $(0, 0, 2), (0, 0, 3), (0, 0, 4), \ldots$ respectively.



The surface $z = x^2 + y^2$ has a bowl shape. It is called an **elliptic paraboloid.**

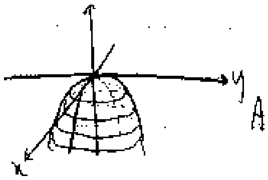
tada! A BOWL (the Elliptic Paraboloid)



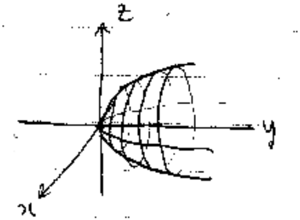
- The general equation of the surface is

$$\pm(z - z_0) = \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2}$$

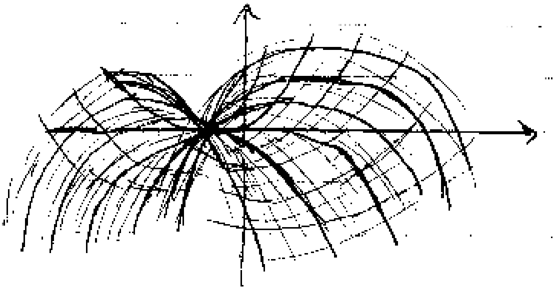
- a and b are non-zero constants that stretch or contract the graph in the x and y directions respectively, and where (x_0, y_0, z_0) is the point at the base of the bowl.
- **Note:** the coefficient of $(z - z_0)$ may be negative, in which case, the bowl is upside-down. The surface $-z = x^2 + y^2$ is an upside-down bowl shape.



- It is possible to change the roles of the x, y and z variables giving the an elliptic paraboloid, which is orientated differently in 3-dimensional space.
- We distinguish amongst the different orientations by noting which axis (x-axis, y-axis, or z-axis) goes through the center of the paraboloid.
- The axis of the elliptic paraboloid $z = x^2 + y^2$ is the z-axis. Also, $x = y^2 + z^2$ is an elliptic paraboloid with axis the x-axis.
- The surface $y = x^2 + z^2$ is an elliptic paraboloid with axis the y-axis



EXAMPLE graph the quadric surface $z = x^2 - y^2$. The surface looks like a horse saddle. It is called a **hyperbolic paraboloid.**



The following table lists the traces given by intersecting the hyperbolic paraboloid $z = x^2 - y^2$ with a plane (c is a constant).

Plane	Trace
$x = c$	parabola : $z = c^2 - y^2$
$y = c$	parabola : $z = x^2 - c^2$
$z = c$	hyperbola: $x^2 - y^2 = c$

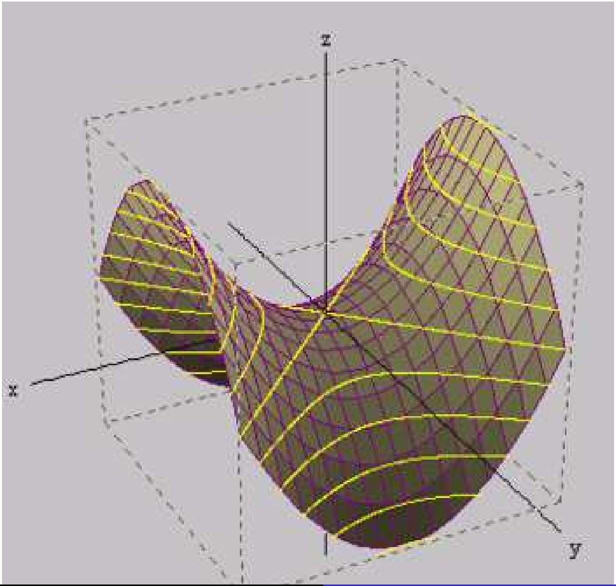
- The general equation of the hyperbolic paraboloid is

$$\pm(z - z_0) = \frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2}$$

  where a and b are scaling factors, and where (x_0, y_0, z_0) is the point at the center of the saddle.

- The main axis of the hyperbolic paraboloid is given by the non-quadratic variable.

tada! A HORSE SADDLE (the Hyperbolic Paraboloid)



The simplest form of an **Ellipsoid** is

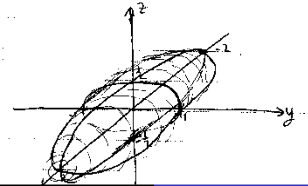
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

. An ellipsoid, a sphere that has been stretched or squeezed in some axis directions.

The traces given by intersecting an with a plane (c is a constant).

Plane	Trace
$x = d$	ellipse: $\frac{z^2}{c^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{a^2}$
$y = d$	ellipse: $\frac{z^2}{c^2} + \frac{x^2}{a^2} = 1 - \frac{d^2}{b^2}$
$z = d$	ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2}$

EXAMPLE: $\frac{x^2}{2^2} + y^2 + \frac{z^2}{\left(\frac{1}{2}\right)^2} = 1$

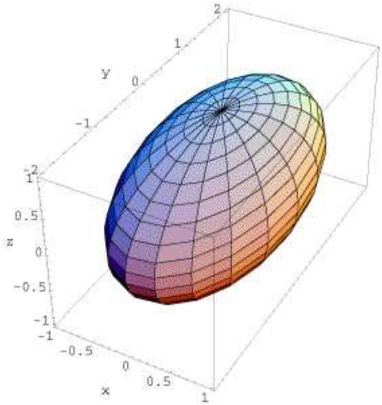


- The general equation of an ellipsoid is

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1$$

  where a, b and c are scaling factors, and where (x_0, y_0, z_0) is the point at the center of the ellipsoid.
- Note that if $a > 1$, $b > 1$ or $c > 1$, then the ellipsoid looks like sphere that has been stretched in the x-axis, y-axis or z-axis direction respectively.
- Similarly, if $a < 1$, $b < 1$ or $c < 1$, then the ellipsoid looks like a sphere that has been squeezed in the x-axis, y-axis or z-axis direction respectively.

tada! A FOOTBALL (Ellipsoid)

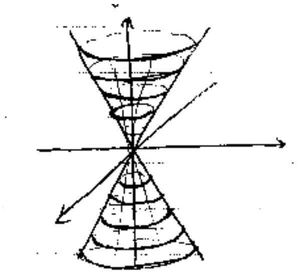


A basic **HYPERBOLOID** has the basic form $x^2 + y^2 - z^2 = c$ where c is some constant.

There are three hyperboloids depending on the value of c:

1. If $c = 0$, **Elliptic cone**;
2. $c = 1$, **Hyperboloid of one sheet**;
3. $c = -1$ **Hyperboloid of 2 sheets**

The basic form of an elliptic cone is $x^2 + y^2 - z^2 = 0$.



Traces given by intersecting with a plane (c is a constant).

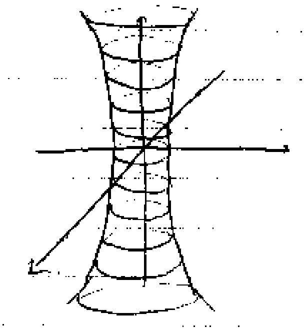
Plane	Trace
$x = c$	hyperbola : $y^2 - z^2 = -c^2$
$y = c$	hyperbola : $x^2 - z^2 = -c^2$
$z = c$	ellipse : $x^2 + y^2 = c^2$

- The axis of the elliptic cone $x^2 + y^2 - z^2 = 0$ is the z-axis, which goes through the center of the cone.
- Note that for $x^2 + z^2 - y^2 = 0$ is an elliptic cone with axis the y-axis, and $y^2 + z^2 - x^2 = 0$ is an elliptic cone with axis the x-axis.
- The general form of an elliptic cone is

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = 0.$$

Hyperboloid of one sheet

The basic form of a hyperboloid of one sheet is $x^2 + y^2 - z^2 = 1$.



Traces given by intersecting with a plane (c is a constant).

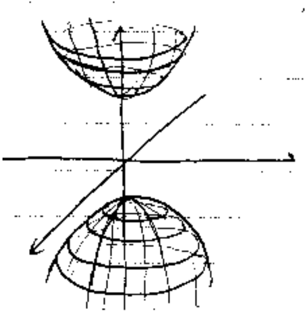
Plane	Trace
$x = c$	hyperbola : $y^2 - z^2 = 1 - c^2$
$y = c$	hyperbola : $x^2 - z^2 = 1 - c^2$
$z = c$	ellipse : $x^2 + y^2 = c^2 + 1$

- The axis of the hyperboloid $x^2 + y^2 - z^2 = 1$ is the z-axis, which goes through the center of the cone.
- Note that for $x^2 + z^2 - y^2 = 1$ is hyperboloid with axis the y-axis, and $y^2 + z^2 - x^2 = 1$ is a hyperboloid with axis the x-axis.
- The general form of a hyperboloid of one sheet is

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = 1.$$

Hyperboloid of two sheets

The basic form of a hyperboloid of two sheets is
$x^2 + y^2 - z^2 = -1$.



Traces given by intersecting the some plane (c is a constant).

Plane	Trace
$x = c$	hyperbola : $y^2 - z^2 = -1 - c^2$
$y = c$	hyperbola : $x^2 - z^2 = -1 - c^2$

- The axis of the hyperboloid $x^2 + y^2 - z^2 = -1$ is the z-axis, which goes through the center of the cone.
- Note that for $x^2 + z^2 - y^2 = -1$ is a hyperboloid with axis the y-axis, and $y^2 + z^2 - x^2 = -1$ is a hyperboloid with axis the x-axis.
- The general form of a hyperboloid of one sheet is

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = -1.$$

Summary of types of quadric surfaces.

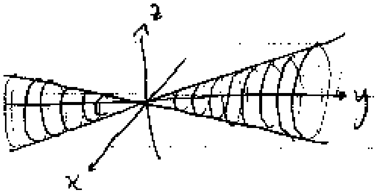
Surface	Type	Basic Equation
Ellipsoid		$x^2 + y^2 + z^2 = 1$
Paraboloid	Elliptic	$z = x^2 + y^2$
	Hyperbolic	$z = x^2 - y^2$
Hyperboloid	Elliptic Cone	$x^2 + y^2 - z^2 = 0$
	one sheet	$x^2 + y^2 - z^2 = 1$
	two sheets	$x^2 + y^2 - z^2 = -1$

Method to recognize surfaces:

When classifying a quadric surface:

1. learn to compute TRACES and to reconstruct the 3-D figure from looking at the traces.
2. Memorize the Summary table and names (Pages: 468-469).
3. decide whether it is an ellipsoid, paraboloid, or hyperboloid. How? From traces and degrees of variables present!!!
4. Then decide what type it is (e.g., elliptic paraboloid?). How? From Traces and signs of non-zero coefficients.
5. what the axis of the surface is (i.e., x-axis, y-axis, or z-axis) (note that an ellipsoid has no axis). How? From traces, signs or degree determine the main axis and orientation.

1. Classify the surface $\frac{x^2}{2} - \frac{y^2}{4} + \frac{z^2}{3^2} = 0$.



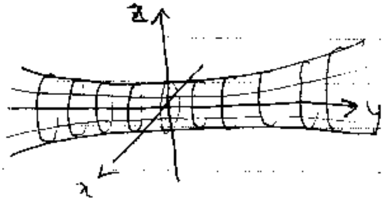
$$\frac{x^2}{2} - \frac{y^2}{4} + \frac{z^2}{3^2} = 0 \Rightarrow \frac{x^2}{2} + \frac{z^2}{3^2} - \frac{y^2}{4} = 0.$$

This surface is an elliptic cone with axis the y-axis. Recall that the axis of the elliptic cone is the axis that goes through its middle and is the variable with a negative sign.

2. Classify the surface $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$.
This surface is an ellipsoid.

PROBLEM:

Classify the surface $4x^2 - y^2 + z^2 - 4z = 0$.



$$4x^2 - y^2 + z^2 - 4z = 0$$

$$\Leftrightarrow 4x^2 - y^2 + (z-2)^2 - 4 = 0$$

$$\Leftrightarrow x^2 - \frac{y^2}{2^2} + \frac{(z-2)^2}{2^2} - 1 = 0$$

$$\Leftrightarrow x^2 + \frac{(z-2)^2}{2^2} - \frac{y^2}{2^2} = 1$$

Surface is a hyperboloid of one sheet with axis the y axis. The axis