1. Explain what is a conservative vector field. Make sure to mention at least two of its properties. Then determine which of the following vector fields is conservative:

   (a) $F_1 = (xe^y)i + (ye^z)j + (ze^x)k$.

   (b) $F_2 = (zcos(xz))i + (e^y)j + (xcos(xz))k$.

Conservative means $F = \nabla G$. This implies work on a closed loop is zero and work from A to B is the same regardless of the trayectory $= (G(B) - G(A)$

(a) $\frac{\partial x e^y}{\partial z} = 0$ but $\frac{\partial z e^x}{\partial x} = z e^x$ thus <u>not</u> conservative

(b) $\frac{\partial z\cos(xz)}{\partial z} = \cos(xz) - zx\sin(xz) = \frac{\partial x\cos(xz)}{\partial x}$

$\frac{\partial z\cos(xz)}{\partial y} = 0 = \frac{\partial e^y}{\partial x}$ and $\frac{\partial e^y}{\partial z} = 0 = \frac{\partial x\cos(xz)}{\partial y}$

$\Rightarrow$ Conservative.

2. Let $A = (0,0)$, $B = (1,0)$, $C = (1,1)$ and $D = (0,1)$, Let σ be the curve made up by the three line segments AB, BC, CD.

(a) Find the work done by a particle traversing σ counterclockwise under the force field $F = x^2 i + y^2 j$.

(b) If a fence is constructed with σ as its base and a height function given by the function $x^2 + 2y$, find the total area of the fence (on one of its sides).

(a) There are several solutions (e.g. direct application of formula and parametrization for 3 curves)

The super-fastest solution

Field is conservative!

$G = \frac{x^3}{3} + \frac{y^3}{3}$ has $\nabla G = x^2 i + y^2 j$

thus work does not matter on the path taken only the start-end points

$$\text{Work} = G(0,1) - G(0,0) = \frac{1}{3}.$$

(b) Formula says area = $\int F(c(t))\,\|c'(t)\|\,dt$

For these calculations we parametrize 3 segments

$AB \to c_{AB} = t\,i \qquad c'_{AB} = i \Rightarrow |c'_{AB}| = 1$

$BC \to c_{BC} = i + t\,j \qquad c'_{BC} = j \qquad |c'_{BC}| = 1$

$CD \to c_{CD} = (1-t)\,i + j \qquad c'_{CD} = -i \qquad |c'_{CD}| = 1$

$$\Rightarrow \underbrace{\int_0^1 t^2\,dt}_{c_{AB}} + \underbrace{\int_0^1 (1+2t)\,dt}_{c_{BC}} + \underbrace{\int_0^1 [(1-t)^2+2]\,dt}_{c_{CD}}$$

$$= \tfrac{1}{3} + 2 + \tfrac{7}{3} = \tfrac{14}{3} \text{ area}$$

3. Let C be the curve $C(t) = (3cos^3(t) + 6)i + (3sin^3(t) + 6)j + \pi k$ from $t = 0$ to $t = \pi/2$.

(a) Find the arc length of C from $t = 0$ to $t = \pi/2$.

(b) For all conservative fields in question 1 find the the work done by a particle that moves from $(9, 6, \pi)$ to $(6, 9, \pi)$ following the curve C.

a) For arc length $= \int_0^{\pi/2} |v(t)| dt = \int_0^{\frac{\pi}{2}} \sqrt{(-9\cos^2 t \sin t)^2 + (9 \sin^2 t \cos t)^2}\, dt$

$= \int_0^{\frac{\pi}{2}} 9 \cos(t) \sin(t)\, dt = 9/2$

(b) To recover potential function $G = \int z \cos(xz)\, dx = \sin(xz) + h(y,z)$

Also $G = \int x \cos(xz)\, dz = \sin(xz) + r(x,y)$ thus

$\frac{\partial G}{\partial z} = x\cos(xz) = x\cos(xz) + \frac{\partial h(y,z)}{\partial z} \Rightarrow \frac{\partial h(y,z)}{\partial z} = 0 \Rightarrow h(y,z)$ has only y variabl

and $\frac{\partial G}{\partial y} = e^y \Rightarrow h(y) = e^y \Rightarrow G = \sin(xz) + e^y$

Thus for work $\int F_2 \cdot dr =$

$= G(6, 9, \pi) - G(9, 6, \pi)$

$= e^9 - e^6$

4. Let C be the closed curve $C(t) = (3cos^3(t)+6)i + (3sin^3(t)+6)j$ for $t = 0$ to 2π.

(a) Find the value of the line integral

$$\int_C ln(x)sin(y)dy - \frac{cos(y)}{x}dx$$

(b) Do you expect the same answer for the line integral if you use $D(t)\bar{}(3cos^3(t))i + (3sin^3(t))j$? for $t = 0$ to 2π? Explain.

(a) We can use the Flux version of Green's theorem

$M = \ln(x)\sin(y)$ $\quad N = \frac{\cos(y)}{x}$ $\quad \frac{\partial M}{\partial x} = \frac{\sin(y)}{x}$

$\frac{\partial N}{\partial y} = -\frac{\sin(y)}{x}$

All functions are continues inside the curve C.

$$\Rightarrow \int_C \ln(x)\sin(y)\,dy - \frac{\cos(y)}{x}\,dx = \iint_C \left(\frac{\sin(y)}{x} + \left(-\frac{\sin(y)}{x}\right)\right) dx\,dy = 0$$

b) No because D(t) contains (0,0) where function is not continuous.