Math 21D De Loera
Final exam
March 3 2008

Name:
Student ID#
Seat Number and Row letter

READ, UNDERSTAND THE INSTRUCTIONS FIRST!

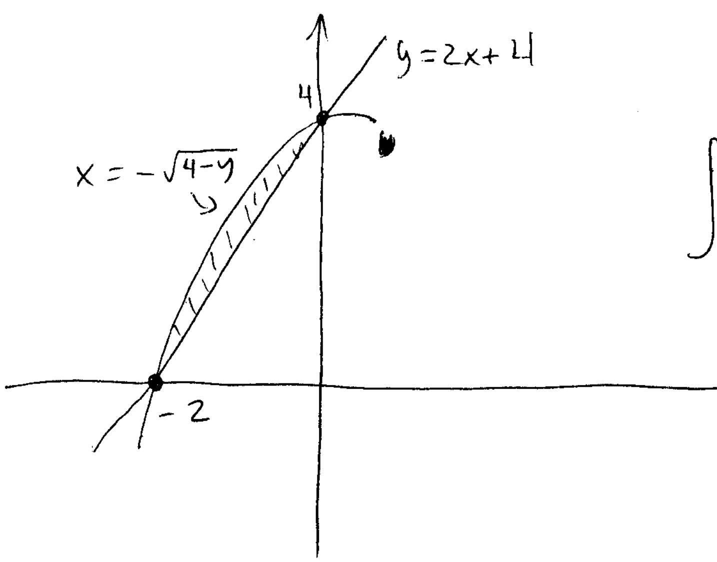
(1)Do NOT turn this page until told to do so!
(2) Fill in above information (your name, etc) NOW!!
(3) Show your work on every problem.
(4) Be organized and clean. Write legibly!
(5) No calculators are allowed, nor is any kind of assistance
(6) Only a pencil and an eraser should be on your desk. No cell phones please!

#	Student's Score	Maximum possible Score
1		5
2		5
3		5
4		4
5		4
6		4
7		6
8		6
9		6
Total points		45

1. Sketch the region of integration and write an equivalent integral with the order of integration reversed for

$$\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx dy.$$

Evaluate one of the integrals.



Thus new integral

$$\int_{-2}^{0} \int_{2x+4}^{4-x^2} dy\, dx$$

$$= \int_{-2}^{0} (-x^2 - 2x)\, dx$$

$$= \left[-\frac{x^3}{3} - x^2 \right]_{-2}^{0} = \frac{4}{3}$$

2. Find the mass of a disk plate centered at the origin bounded by the circle $x^2 + y^2 = 1$ and whose density is given by the function $f(x, y) = ln(x^2 + y^2 + 1)$. Where is the center of mass?

$$\text{Mass} = \iint_{\text{REGION}} (\text{density})\, dR = \int_0^{2\pi} \int_0^1 \ln(r^2+1)\, r\, dr\, d\theta$$

$$u = r^2 + 1$$

$$= \int_0^{2\pi} \left(\frac{1}{2} \int_1^2 \ln(u)\, du \right) d\theta = \int_0^{2\pi} \frac{1}{2} \left[u \ln u - u \right] \Big|_1^2 \, d\theta$$

$$= (\ln(4) - 1)\, \pi$$

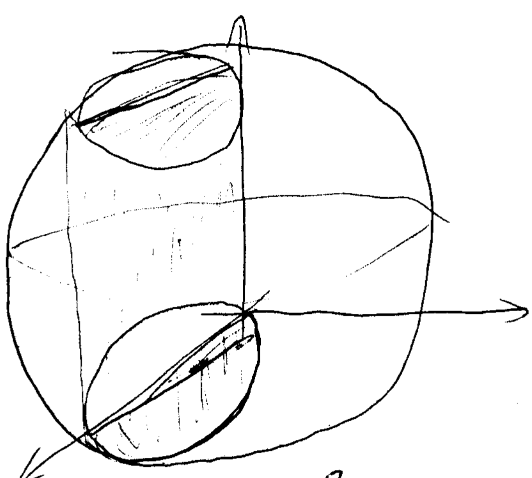
3. The volume of a solid is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

(a) Describe the solid by giving equations for the surfaces that form its boundary. Make sure to use words and a picture to give a clear description. (b) Convert the integral to cylindrical coordinates. Do NOT evaluate the integrals!

(a) $z = \pm\sqrt{4-x^2-y^2} \Rightarrow x^2+y^2+z^2=4$ thus region is bonded on top and bottom by sphere

$y=0$, $y=\sqrt{2x-x^2}$ the first gives a plane (xz-plane) the other is $x^2+y^2-2x=0 \Rightarrow (x-1)^2+y^2=1$



So left ~~is~~ a plane $y=0$ bounding right a cylinder. Thus body look like half-cylinder with sphere-like top-bottom.

(b) $$\int_0^{\pi/2} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\,dr\,d\theta$$

4. (a) Given an intuitive explanation of what is a line integral. (b) Integrate the function $f(x,y,z) = \sqrt{x^2+y^2}$ over the curve $r(t) = (cos(t) - tsin(t))i + (sin(t) - tcos(t))j$, with $0 \le t \le \sqrt{3}$.

(a) The line integral $\int_C f(x,y,z)\,dC$ can be interpreted in many useful ways, including work of particle moving along C agants a force. But the simplest idea is that of surface area of a fence constructed over the base curve C with height $f(x,y,z)$ over point (x,y,z)

(b) Let us set up the integral

$$\int_r \sqrt{x^2(t)+y(t)^2}\,dr = \int_0^{\sqrt{3}} \sqrt{x^2(t)+y^2(t)}\,|r'(t)|\,dt$$

$$\sqrt{x^2(t)+y^2(t)} = \sqrt{(\cos t - t\sin t)^2 + (\sin t - t\cos t)^2}$$

$$= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 + \sin^2 t - 2t\sin t\cos t + t^2\cos^2 t}$$

$$= \sqrt{1 - 4t\cos t\sin t + t^2}$$

$$r'(t) = (-2\sin t - t\cos t)\,i + (t\sin t)\,j$$

$$|r'(t)| = \sqrt{(-2\sin(t) - t\cos(t))^2 + t^2\sin^2 t}$$

$$= \sqrt{4 + 4t\sin(t)\cos(t) - 4\cos^2(t) + t^2}$$

$$\int_0^{\sqrt{3}} \sqrt{1-4t\cos t\sin t + t^2}\,\sqrt{4+t^2+4t\sin t\cos t - 4\cos^2 t}\,dt$$

5. (a) State Green's theorem.

(b) Evaluate the integral

$$\oint_C (2x + y^2)dx + (2xy + 3y)dy$$

where C is the circle $(x-2)^2 + (y-3)^2 = 4$.

(a) Green's theorem is about how a line integral can be turned into a double integral or (vice versa). In its circulation form say For a region R bounded by a closed curve C

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dR$$

$$\frac{\partial N}{\partial x} = 2y \qquad \frac{\partial M}{\partial y} = 2y$$

$$\iint (2y - 2y)\, dx\, dy = 0 = \oint_C (2x + y^2) dx + (2xy + 3y) dy$$

6. Find a parametrization for the cone $z = 1 + \sqrt{x^2 + y^2}$, for $z \leq 3$.

If we say $x = r\cos\theta$, $y = r\sin\theta$

$$z = 1 + \sqrt{r^2\cos^2 + r^2\sin^2\theta} = 1 + r$$

Thus the surface of the cone is given by

$(r\cos\theta)\,i + (r\sin\theta)\,j + (1+r)\,k$ with $0 \leq \theta \leq 2\pi$

now $0 \leq z \leq 3 \Rightarrow -1 \leq r \leq 2$

7. (a) Find the area of the surface $S(u,v) = (u+v)i + (u-v)j + vk$, for $0 \le u \le 1$, $0 \le v \le 1$. (b) Find the average value of the function $f(x,y,z) = xy - z^2$ over the surface S above.

(a) Using the parametrization

$T_u = i + j$, $T_v = i - j + k$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = i - j - 2k \Rightarrow |T_u \times T_v| = \sqrt{6}$$

Thus surface area integral $= \iint_{\text{Region}} |T_u \times T_v| \, du \, dv = \int_0^1 \int_0^1 \sqrt{6} \, du \, dv$

$= \sqrt{6}$

(b) Now $\iint_S (xy - z^2) \, d\sigma = \int_0^1 \int_0^1 [(u+v)(u-v) - v^2] \sqrt{6} \, du \, dv$

$= \sqrt{6} \int_0^1 \int_0^1 (u^2 - 2v^2) \, du \, dv = \sqrt{6} \int_0^1 \left[\frac{u^3}{3} - 2uv^2\right]_0^1 dv$

$= \sqrt{6} \int_0^1 \left(\frac{1}{3} - 2v^2\right) dv = \underline{-\frac{\sqrt{6}}{3}}$

8. (a) Find the moment of inertia about the z-axis of the **solid** box cut from the first octant by the planes $x = a, y = b$, and $z = c$. Assume density is constant 1. (b) Let $F = (2xy)i + (2yz)j + (2xz)k$ be a vector field. Set up the integral that expresses the outward flux of $CURL(F)$ accross the top square (at $z = c$) of the box in part (a). Do not evaluate the integral!

(a)

$$I_z = \int_0^a \int_0^b \int_0^c (x^2+y^2)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b (x^2+y^2)\, z \Big|_0^c dy\, dx$$

$$= \int_0^a \int_0^b (x^2+y^2)\, c\, dy\, dx$$

$$= \int_0^a \left(c\, x^2 y + \frac{y^3}{3} \Big|_0^b \right) dx$$

$$= \int_0^a \left(c\, b\, x^2 + \frac{cb^3}{3} \right) dx$$

$$= \int_0^a c b \frac{x^3}{3} + \frac{cb^3 x}{3} \Big|_0^a = \frac{cba^3}{3} + \frac{cb^3 a}{3}$$

$$= abc \left(\frac{a^2+b^2}{3} \right)$$

(b)

8. (a) Find the moment of inertia about the z-axis of the **solid** box cut from the first octant by the planes $x = a$,$y = b$, and $z = c$. Assume density is constant 1. (b) Let $F = (2xy)i + (2yz)j + (2xz)k$ be a vector field. Set up the integral that expresses the outward flux of $CURL(F)$ accross the top square (at $z = c$) of the box in part (a). Do not evaluate the integral!

(b) $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & 2yz & 2xz \end{vmatrix}$

$= i(0-2y) - j(2z-0) + k(0-2x) = (-2y)i - 2zj - 2xk$

Flux $\iint_S (\nabla \times F \cdot n)\, d\sigma$

surface is at plane $z = c \leftarrow$ Equation f

thus $n = k$

$d\sigma = \frac{|\nabla f|\, dR}{|\nabla f \cdot k|} = \frac{1}{1} dR$

$\iint_R (-2x)\, dR = \int_0^a \int_0^b -2x\, dy\, dx = \int_0^a (-2x\, b)\, dx$

$= \int_0^a -2x\, b\, dx = -bx^2 \Big|_0^a = \boxed{-ba^2}$

9. (a) State Stoke's theorem. (b) Given the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, verify Stokes' Theorem for the part of the surface $z = 1 - x^2 - y^2$, with outward unit normal $\mathbf{n}$, that lies above the xy-plane.

(a) Stokes theorem says that for a surface M with boundary curve C, oriented counterclockwise

$$\oint_C F \cdot dC = \iint_M (\nabla \times F) \cdot n \, d\sigma$$

(b) The surface is a paraboloid with boundary curve a circle $x^2+y^2=1$ (the intersection of the plane $z=0$ with $z=1-x^2-y^2$)

Now
$$(\nabla \times F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = i(0) - j(0) + k(1-(-1$$
$$= 2k$$

We can find n as normal to the surface

$z = 1 - x^2 - y^2$ $\quad \nabla f = (2x)i + (2y)j + 1 \cdot k$

$|\nabla f \cdot k| = 1$, $\quad |\nabla f| = \sqrt{4x^2+4y^2+1} = .$

$$n = \frac{\nabla f}{|\nabla f|} \qquad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|}$$

$$\iint_M \nabla \times F \cdot n \, d\sigma = \iint_M (\nabla f) \cdot 2k \, dR = \iint_{\substack{\text{circle} \\ \text{of radius 1}}} 2 \, dR = 2\pi$$

On the other hand C is circle $x = \cos t$
$y = \sin t$.

$$\oint_C M\,dx + N\,dy = \int_C -y\,dx + x\,dy$$

$$= \int_0^{2\pi} (-\sin t)(-\sin t) + \cos t\,(\cos t)\,dt$$

$$= \int_0^{2\pi} dt = 2\pi \text{ as predicted}$$