1. (5 points) Find a basis for the nullspace of the matrix (as always, give all details):

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \end{bmatrix}$$

Reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 0 & 5/2 \end{bmatrix}.$$ This means that

a basis for nullspace is $= \left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 \\ 5/2 \\ 0 \\ 1 \end{pmatrix} \right\}$

2. (6 points) State the definition of a set of linear independent vectors. For what values of λ is the set of vectors $\{(\lambda^2-5,1,0),(2,-2,3),(2,3,-3)\}$ linearly independent? For what values is it a basis for R^3? Give details! Answer the same questions for the following set of vectors in R^2: $\{(a^2,3),(-1,a-2),(a,a),(-a^3-2a+2,22),(a/2,-1/a)\}$.

In the first case;
The 3 vectors for a basis for $\mathbb{R}^3$ precisely when matrix is invertible.

$$\det\begin{pmatrix} \lambda^2-5 & 1 & 0 \\ 2 & -2 & 3 \\ 2 & 3 & -3 \end{pmatrix} = -3\lambda^2+27$$

This becomes zero only when $\lambda = 3, -3$, therefore The set is a basis for $\mathbb{R}^3$ for all λ <u>except</u> $3, -3$.

In the second case: 5 vectors in $\mathbb{R}^2$ are ALWAYS Linearly dependent, thus they would <u>never</u> form a basis.

3. (5 points) Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 2 & 1 & -2 & 0 & 1 \\ 2 & 2 & 0 & 0 & 1 \end{bmatrix}$$

and determine the dimension of its nullspace. Give details!

Reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -6 & -\frac{1}{2} \\ 0 & 1 & 0 & 6 & 1 \\ 0 & 0 & 1 & -3 & -\frac{1}{2} \end{bmatrix}$

Thus $\text{rank}(A) = 3$

Since $\text{rank}(A) + \text{dimension}(\text{nullspace}(A)) = 5$

$\Rightarrow$ dimension of nullspace$(A) = 2$.

4. (4 points) Decide whether the following statements are true or false (give a short justification if you want full points!):

(a) Let A be an $m \times n$ matrix. The set of vectors x such that $Ax \neq 0$ is a subspace of R^n.

(b) Every linearly independent set of vectors in R^{17} contains seventeen vectors.

(c) The nullspace of matrix A is spanned by the columns of A.

(d) If A is an 8×8 singular matrix, then $rank(A) < 8$.

a) False, $\bar{0}$ does not belong to this set therefore it is not a subspace.

b) False, $(1, \overbrace{0, 0, 0, \dots 0}^{16 \text{ times}})$ $(0, 1, \underbrace{0, \dots\dots 0}_{15 \text{ times}})$ are linearly independent.

c) False, see for instance matrix in exercise 3. Column vectors live in $\mathbb{R}^3$, but nullspace is a subspace of $\mathbb{R}^5$

d) True, $\text{rank}(A) = 8 \iff 8 \times 8$ matrix is non-singular.