The strongest shape: Moduli theory through the lens of triangles
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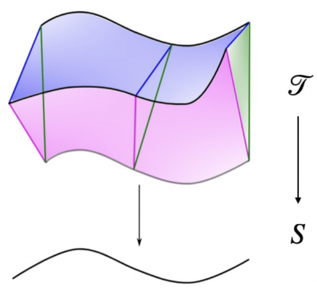
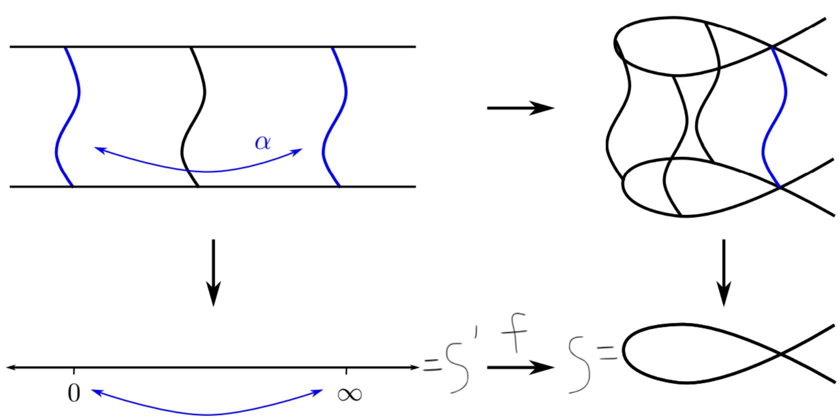
0.1 Motivation

In geometry, the notion of deformation is a standard thing to study. A deformation is, for us, nothing but just a *family* of objects parametrized over some indexing *space*. This seminar is *not* about deformation theory, but rather we wish to use it as a motivation of a philosophy: family of objects are more interesting than just a single object on its own. The study of families of geometric objects is the so-called **moduli theory**. And as the title suggests, we will take **triangles** as our geometric objects.



Mathematicians are often very greedy, so let's consider the family of *all* triangles. One should visualize this family, denoted $\mathcal{M} \to M$, as some **universal family** just like the one in the diagram above. It is *universal* in the sense that every triangle can be obtained as a "slice" of $\mathcal{M}$.

Now consider a continuous map $f : S' \to S$. Suppose a family of triangles $\mathcal{F} \to S$ is given on S (the right vertical map in the diagram below). Then, from these data, we can naturally associate a family on S' (the left vertical map in the diagram below):



(Here imagine all the squiggly lines are triangles.) This process is called **pullback**.

In particular, this construction applies to the family $\mathcal{M} \to M$ (in place of the family $\mathcal{F} \to S$). Since $\mathcal{M}$ contains all the triangles, it is natural to ask about the converse:

> **Question**: Does every family of triangles, parametrized over a fixed base space S, appears from a continuous map $S \to M$?

As it turns out, the answer is almost always ... *no*! The obstruction here is both very surprising and simple: it is the symmetries that some triangles possess.

We can fix this in two fundamentally different ways.

(1) Impose more and more tructures to the triangles and requires all symmetries to preserve them, until you kill all the nontrivial symmetries. (This process is called **rigidification**.)

(2) Try to remember *how* triangles are equivalent, not just *when*.

This latter approach leads to the abstract concept of **moduli stacks**.

0.2 Program descriptions and expectations

The goal for this seminar is to basically trying to make some sense on the technical detail we have omitted in the discussion above. This seminar has two parts.

In the first few weeks, students will acquaint themselves with the concept of categories. The language of categories is a beautiful invention allowing one to adequately package the technical detail. (Though it offers not much insight on new discovery.)

After that, we will dive into the world of triangles, guided with a bunch of beautifully drawn pictures in the reading. We will *not* pursue the language of stacks, as that requires, first and foremost, a deeper dive into category theory, and one can't justify the need for stacks until they have acquired enough geometric intuition. Because of that, *the two parts of this seminar are almost independent of each other.*

Students are expected to do the assigned reading and come to our weekly meeting to have a chat on their progress. We expect students to have at least one question ready for each meeting regarding the material. Some exercises will be assigned along the way. We don't expect the students to be able to do all the assigned exercises, but rather they should take time to think about them, make sense of them, and (try to) understand what might be needed to solve them.

0.3 Prerequisites and Readings

We ask the participants to have at least taken MAT 150A. We will assume the participants to have some exposure to topology. MAT 147 is more than enough, but MAT 127A is also good enough (though one might need to look up a few terms occasionally).

(1) For the foundation of category theory, we shall consult Chapter 1 of

Leinster, T. (2016). *Basic category theory.* arXiv preprint arXiv:1612.09375.

(2) Our main reading material will be the Chapter 1 of the lecture note

Behrend, K. (2014). *Introduction to algebraic stacks.*

(available at https://personal.math.ubc.ca/~behrend/math615A/stacksintro.pdf)

Besides, there are also some short supplemental readings.

(3) John Baez (from UC Riverside) has an excellent blog post about acute triangles, as well as their connection to elliptic curves: `https://golem.ph.utexas.edu/category/2023/09/the_moduli_space_of_acute_tria.html`

(4) Someone suggested to me the following 15-page paper, which we might take some time to have a look:

Brussel, E., & Goertz, M. E. (2023). *The Torus of Triangles.* arXiv preprint arXiv:2303.11446.

0.4 Program outline

The program is tentatively divided as follows:

- Meeting 1-3: Categories, functors, and natural transformations.
- Meeting 4: Families of triangles and symmetry groupoids.
- Meeting 5: Coarse moduli space of all triangles, pullback of families, and moduli map.
- Meeting 6: Scalene, isosceles, and equilateral triangles, and their moduli spaces
- Meeting 7: Oriented triangles.
- Meeting 8-9: (TBD) Degenerated triangles perhaps.