


Cherednik algebras, Hilbert scheme and mirabolic D-modules.

Notations:

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$$\mathfrak{g} = \mathfrak{gl}_n \quad n \geq 2 \quad \mathfrak{g}^k \cong \mathfrak{g}^* \quad G = GL_n$$

U

\mathfrak{f} Borel, upper triangular

U

B

U

\mathfrak{t} Cartan, diagonal

U

T

$W = S_n$, Weyl group

$$\text{Fix } C = \frac{m}{n}, \quad m \in \mathbb{Z}_{>0}, \quad (m, n) = 1.$$

Cherednik algebra

H_C : C -algebra generated by $\mathfrak{h}, \mathfrak{h}^*, w$
with relations

$$\left\{ \begin{array}{l} [x, x'] = 0, \quad [y, y'] = 0, \\ w \cdot x \cdot w^{-1} = w(x), \quad w \cdot y \cdot w^{-1} = w(y) \\ [y, x] = x(y) - \sum_{s \in S} c(\alpha_s, y) \langle \alpha_s^\vee, x \rangle s \end{array} \right\}$$

$s \in S \subseteq W$, reflections

$x, x' \in h^*$, $y, y' \in h$, $w \in W$,
 α_s : simple root associated to s
 α_s^\vee simple coroot associated to s

Quantization of $(\mathbb{C}[h \times h^*])^W$.

$$e = \frac{1}{|W|} \sum_{w \in W} w \quad A_c := e H_c e \quad \text{spherical Cherednik algebra}$$

Quantization of $(\mathbb{C}[h \times h^*])^W$

$$A_c \dashrightarrow \mathcal{G}^r \dashrightarrow h \times h^* // W \supseteq \{0\}$$

\uparrow Hilbert-Chow \uparrow

$$\dashrightarrow \text{Hilb}^n(\mathbb{C}^2) =: \text{Hilb} \leftarrow \text{Hilb}_0.$$

Hilbert scheme of n points on the plane.

Gordon-Stafford 05'

$$GS : A_c\text{-Fmod} \rightarrow \text{Coh}(\text{Hilb})$$

\hookdownarrow filtered.

$\mathbb{C}[h] \cong H_c \otimes_{(\mathbb{C}[h^*])^W} \mathbb{C}$ has a max proper sub
 with quotient L_c

\downarrow

Thm (Berest, Etingof, Ginzburg '03)

eL_c is the unique irreducible finite dimensional representation of A_c .

$GS(eL_c) = ?$

Thm ('06, G-S).
 $m = nk + 1, k \in \mathbb{Z}_{\geq 0}$.

$GS(eL_c) = \bigcup_{\text{Hilb}_c} (k).$

General $m \neq ?$

$$V = \mathbb{C}^n$$

$\widetilde{\text{Hilb}} = \{(x, y, v) \in \mathcal{J}_X \times V \mid [x, y] = 0, ([x, y]v) = V\}$

$\mathcal{J}_G \left[\begin{array}{c} \widetilde{\text{Hilb}} // G = \text{Hilb}_c \\ \widetilde{\text{Hilb}} // G = \text{Hilb}_c \end{array} \right]$

$\widetilde{\text{Hilb}} \supseteq \widetilde{\text{Hilb}}_c \supseteq \mathcal{J}_r.$

$$U \cap \mathcal{J}_r \subseteq \mathcal{J}_r$$

$$\mathcal{L}_A.$$

$\text{codim } \widetilde{\text{Hilb}}_c \setminus \mathcal{J}_r \leq 1.$

$$\text{codim } \widetilde{\text{Hilb}}_c \setminus r\mathcal{J}_r \geq 2.$$

$$G/B.$$

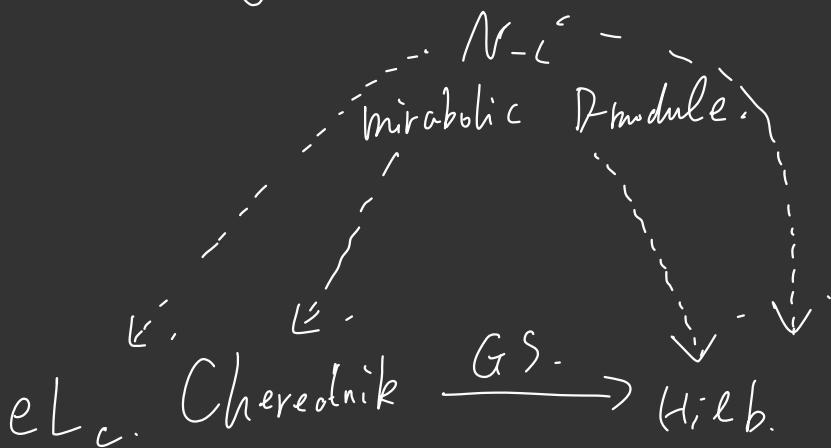
$$G_{Hilb_c}(k) = \text{Desc} \left(j_* \pi^* \mathcal{L}_{ck} \right)$$

$$(k) = (\underbrace{k \dots k}_n).$$

Thm: [m] $G_S(eL_c) = \text{Desc} \left(j_* \pi^* \mathcal{L}_m \right).$

$$m = ([c], [2c]-[c], \dots, m-[(n-1)c]).$$

Gorsky - Negut



Mirabolic D -modules. \mathbb{G} .
 $\mathcal{X} = \mathcal{J} \times V$. V : std.

$D(\mathcal{X}) := P(\mathcal{X}, D_{\mathcal{X}})$. cliff op on \mathcal{X} .

$\tau: \mathcal{J} \hookrightarrow D(\mathcal{X})$.

$T_d: \mathcal{J} \hookrightarrow D(\mathcal{X})$. $x \mapsto \tau(x) - d \text{tr}(x)$.

Hausch-Chandra isom.

Thm [Gan-Ginzburg 2006].

$\forall d \in \mathbb{C}, \quad \left(D(\mathcal{X}) / D(\mathcal{X}) T_d(\mathcal{J}) \right)^{\mathbb{G}} \cong A_{d-1}$.

\leadsto (Hamiltonian reduction functor).

$H_d: (D(\mathcal{X}), \mathcal{J})\text{-mod} \xrightarrow{f.g., ss} A_{d-1}\text{-mod}^{f.g.}$

$\bigcup \mathcal{M} \longmapsto \mathcal{M}^{T_d(\mathcal{J})} \bigcup \mathcal{O}(A_{d-1})$
 \downarrow
 $\mathcal{O}L_{d-1}$.

Def: A regular f.g $D(\mathbb{X})$ -module M is
mirabolic if

$$SS(M) \subseteq \{(x, y, i, j) \in \mathbb{G} \times M \times T^*V \mid [x, y] + i[j] = 0\}$$

$$\overset{\cup}{\widetilde{\text{Hilb}}}_x \xrightarrow{G} \text{Hilb}_x \rightarrow \mathcal{O}_{\mathbb{X}\{\bullet\}/W}$$

\sim Character sheaves.

mirabolic D -module.

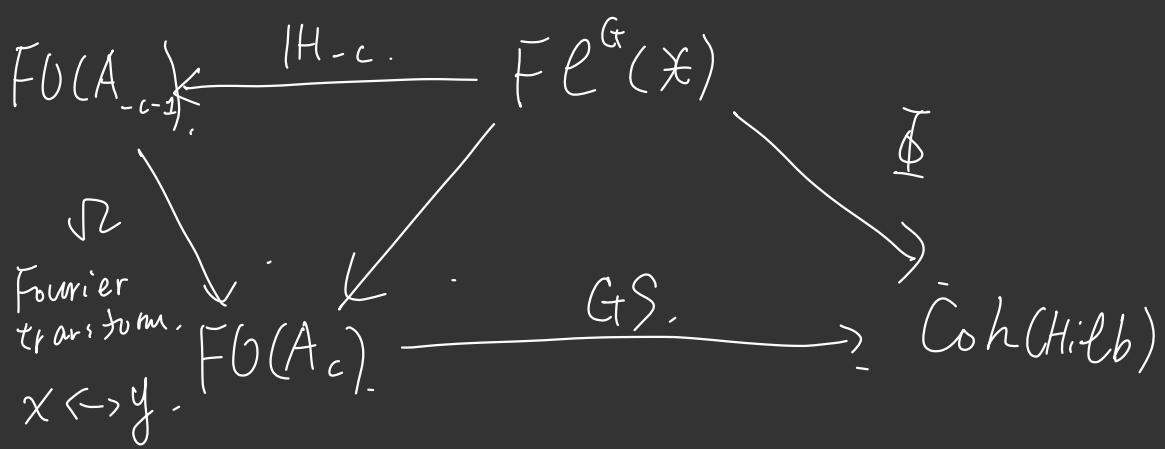
$$(H_d: \mathcal{C}(\mathbb{X}) / \ker(H_d) \xrightarrow{\sim} \mathcal{O}(A_{d-1}))$$

$F\mathcal{E}^G(\mathbb{X})$: G -equivariant filtered, ^(good filtration) mirabolic D -modules.

Descent functor.

$$\mathbb{D}: F\mathcal{E}^G(\mathbb{X}) \rightarrow \text{Coh}(\text{Hilb}).$$

$$(\mathfrak{M}, \underline{F_m}) \mapsto \text{Desc} \left(\text{gr} \mathfrak{M} \Big| \widetilde{\text{Hilb}} \right).$$



$$H_c \leftrightarrow H_{-c}$$

$$\mathcal{U} \qquad \mathcal{L}.$$

$$A_c \rightarrow A_{-c-1}.$$

Thm [M.]. For $(M, F_m) \in \mathcal{F}\ell^G(\mathbb{X})$ s.t

$$\bullet \quad M^G = M^{T_{-c}(\mathfrak{g})}.$$

$\bullet \quad F_m \sim \text{order filtration.}$

$$\left\{ \begin{array}{l} \bullet \quad \mathcal{J}(M, F_m) \subseteq \text{GS} \circ \mathcal{R} \circ H_{-c}(M, F_m). \end{array} \right.$$

$$\text{Rem: } M \in D(\mathbb{X}) / I.$$

Cuspidal mirabolic D-module.

$S: \mathcal{H} \rightarrow \mathbb{C}$.

$(x, v) \mapsto \langle v \circ x, v \wedge x \wedge \dots \wedge x^{n-1} v \rangle$

e.g.: $x = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$S(x, v) = \det \begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{pmatrix}$

Vandermonde.

e.g.: $x = \begin{pmatrix} 0 & x_1 & & * \\ 0 & \ddots & \ddots & \\ 0 & & \ddots & x_{n-1} \\ 0 & & & 0 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$S(x, v) = x_1 x_2^2 \cdots x_{n-1}^{n-1} v_n^n$.

G. $\hookrightarrow Y = \left\{ (x, v) \in \mathcal{H} \times V \mid \begin{array}{l} S(x, v) \neq 0 \\ \text{nilpotent} \end{array} \right\}$.

free, transitive

$$\cdot g \cdot S = \det(g) \cdot S. \quad S \sim \det.$$

$\mathbb{C} \cdot S^{-c}$. Local system on \mathcal{Y} .

Def: Cuspidal metabolic D-module!

N_c is the minimal extension of $\mathbb{C} \cdot S^{-c}$ to \mathcal{X} .

$N_c \sim$ cuspidal character sheaf by Laszlo.

Thm ([Caraque, Enriquez, Etingof '09]).

$$\bigoplus H_{-c}(N_c) = eL_c$$

Goal: . G -equivariant structure

. Hodge filtration,

Restrict to \mathbb{P}_+ , $n = [\ell, \ell]$.

$$Y_0 := Y \cap (n \times V) = \left\{ S_0 = x_1 x_2 \cdots x_{n-1}^{n-1} v_n \neq 0 \right\},$$

$Z := (n \times V) \setminus Y_0$. SNC divisor.

$$\mathbb{C}^* \hookrightarrow \mathbb{C}$$

$L_0 := \mathbb{C} \cdot S_0^{-c}$. Local system on Y_0
 $\downarrow i'$

B. $L := i_! * L_0 = D / \left(\sum D(x_i) \partial_{x_i} + i c \right) + \sum D \partial_{v_j}$

$$\cup \quad D := D(n \times V)$$

$L^{>-1}$ Deligne's canonical extension.

• Locally free, extending L_0 .

• Lattice $L^{>-1} \otimes G_{n \times V}(Z) = L$

• eigenvalues of residues along $Z \subseteq (-1, 0)$
(log connection)

$$B \supset L^{>-1} = \bigcup_{n \times V} [S_0^{-c}]^{-1}.$$

$$S_0^{-c} = x_1^{-c} x_2^{-2c} \cdots x_{n-1}^{-(n-1)c} v_n^{-m}.$$

$$[S_0^{-c}] = x_1^{-[c]} x_2^{-[2c]} \cdots x_{n-1}^{-[(n-1)c]} v_n^{-m}.$$

$$S_0^{-c} [S_0^{-c}]^{-1} \quad \checkmark$$

$$F^k L^{>-1} = L^{>-1} \cap i_* F^k L_0 = \begin{cases} L^{>-1} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

$$F^k L = \sum F_{ord.}^i D F^{k-i} L^{>-1}$$

$$gr_F L = \bigcup_{SS(L)} [S_0^{-c}]^{-1}.$$

$$B \supset [S_0^{-c}]^{-1}.$$

$$\downarrow$$

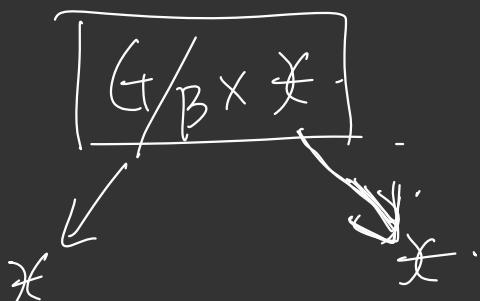
$$B/\langle B, B \rangle \cong T.$$

$$-M = -([c], [2c] - [c], \dots, m - [(n-1)c])$$

Go to N_{-c} .

- Induction of MHM. ([Archarr])
- Lazars' formula for D-module pushforward
- Fourier transform for MHM
([Chen-Dirks., 2021])
 $\mathbb{F}(N_{-c}) \cong N_{-c}$.
- $\text{gr } N_{-c}$ is $(M \text{ [Saito]})$.

$$\text{gr } N_{-c} \Big|_{\tilde{\text{Hilb}}} = j_* \pi^* L_u$$
$$j \uparrow$$
$$r \circ r \xrightarrow{\pi} G/B.$$



L lives on $\mathcal{F} \times V$.
 \downarrow .

$$\mathcal{M}\mathcal{H}\mathcal{M}_B(\mathcal{X}) \xrightarrow{\text{Ind.}_B^G} \mathcal{M}\mathcal{H}\mathcal{M}_G(G \times^B \mathcal{X})$$

\downarrow pushforward

$$\mathcal{M}\mathcal{H}\mathcal{M}_G(\mathcal{X})$$

$$T^*(G \times^B \mathcal{X}) \xleftarrow{f^*} G \times T^*\mathcal{X} \xrightarrow{\omega} T^*\mathcal{X}$$

$$\text{gr}(p_f^* \mathcal{M}) = R\bar{\omega}_* \left(L f^* \text{gr}\mathcal{M} \otimes \omega \right)$$

$$SS(M) \Bigg|_{\mathcal{Z}_r}$$