Cherednik algebras, Hilbert scheme and mirabolic D-modules.

Notations: $/\mathbb{C}$

$\mathfrak{g} = \mathfrak{gl}_n \quad n \geq 2 \quad \mathfrak{g} \overset{K}{\simeq} \mathfrak{g}^* \qquad G = GL_n$

$\cup\!\!\mid \qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\mid$

$\mathfrak{b}$ Borel, upper triangular $\qquad B$

$\cup\!\!\mid \qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\mid$

$\mathfrak{h}$ Cartan, diagonal $\qquad T$

$W = S_n$, Weyl group

Fix $c = \frac{m}{n}$, $\quad m \in \mathbb{Z}_{>0}$, $\quad (m, n) = 1$.

Cherednik algebra

H_c : $\mathbb{C}$-algebra generated by $\mathfrak{h}, \mathfrak{h}^*, W$

with relations

$$\left\{ \begin{array}{l} [x, x'] = 0, \quad [y, y'] = 0, \\ w x w^{-1} = w(x) \quad w y w^{-1} = w(y) \\ [y, x] = x(y) - \sum\limits_{s \in S \subseteq W,\ \text{reflections}} c \langle \alpha_s, y \rangle \langle \alpha_s^\vee, x \rangle s \end{array} \right\}$$

($x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $w \in W$,
α_s : simple root associated to s
$\alpha_s^\vee$ simple coroot associated to s)

Quantization of $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]\rtimes W$.

$e = \frac{1}{|W|}\sum_{w\in W} w$ $\quad A_c := eH_ce$ spherical Cherednik algebra

Quantization of $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^W$

$$\begin{tikzcd} A_c \arrow[r, dashed, "gr"] \arrow[dr, dashed] & \mathfrak{h}\times\mathfrak{h}^*/\!/W & \supseteq \{0\} \\ & \mathrm{Hilb}^n(\mathbb{C}^2) =: \mathrm{Hilb} \arrow[u, "\text{Hilbert-Chow}"'] & \mathrm{Hilb}_0 \arrow[l] \arrow[u] \end{tikzcd}$$

Hilbert scheme of n points on the plane.

Gordon-Stafford 05'

GS : A_c - Fmod $\longrightarrow$ Coh(Hilb)
$\hookrightarrow$ filtered.

$\mathbb{C}[\mathfrak{h}] \underset{\Delta}{\cong} H_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W} \mathbb{C}$ has a max proper sub with quotient L_c

$\downarrow$

L_c

Thm (Berest, Etingof, Ginzburg '03)

eL_c is the unique irreducible finite dimensional representation of A_c.

$GS(eL_c) = ?$

Thm ('06, G-S).

$m = nk+1$, $k \in \mathbb{Z}_{\geq 0}$.

$$GS(eL_c) = \mathcal{O}_{\mathrm{Hilb}_0}(k).$$

General m?

$$V = \mathbb{C}^n$$

$$\widetilde{\mathrm{Hilb}} = \{(x,y,v) \in \underset{\mathcal{N}\times\mathcal{N}}{\mathfrak{g}\times\mathfrak{g}}\times V \mid [x,y]=0,\ \mathbb{C}[x,y]v = V\}$$

$\circlearrowleft$ G

$$\boxed{\widetilde{\mathrm{Hilb}}/\!/G = \mathrm{Hilb}} \qquad \widetilde{\mathrm{Hilb}}_0/\!/G = \mathrm{Hilb}_0$$

$$\begin{tikzcd}
\widetilde{\mathrm{Hilb}} \arrow[r, phantom, "\supseteq"] & \widetilde{\mathrm{Hilb}}_0 \arrow[r, phantom, "\supseteq"] & \mathfrak{Z}_r & & \mathcal{L}_\lambda \arrow[d, dash] \\
& {}_r\mathfrak{Z} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subseteq"] & {}_r\mathfrak{Z}_r \arrow[u, phantom, "\cap"] \arrow[ul, bend left, "j"] \arrow[rr] & & G/B
\end{tikzcd}$$

${}_r\mathfrak{Z}_r \simeq \mathfrak{Z}_r \cup {}_r\mathfrak{Z}$

codim $\widetilde{\mathrm{Hilb}}_0 \setminus \mathfrak{Z}_r = 1$.

codim $\widetilde{\mathrm{Hilb}}_0 \setminus {}_r\mathfrak{Z}_r \geq 2$.

$$\mathcal{O}_{\mathrm{Hilb}_0}(k) = \mathrm{Desc}\left(j_* \pi^* \mathcal{L}_{(k)} \right)$$

$$(k) = (\underbrace{k \cdots k}_{n}).$$

Thm: [M] $GS(eL_c) = \mathrm{Desc}(j_* \pi^* \mathcal{L}_\mu).$

$$\mu = ([c], [2c]-[c], \ldots, m-[(n-1)c]).$$

Gorsky - Negut

$$\begin{tikzcd}
& N_{-c} \ \text{mirabolic } \mathcal{D}\text{-module} \arrow[dl, dashed] \arrow[dr, dashed] & \\
eL_c \ \text{Cherednik} \arrow[rr, "GS"] & & \mathrm{Hilb}
\end{tikzcd}$$

Mirabolic D-modules

$G \circlearrowright \mathfrak{X} = \mathfrak{g} \times V.$ V: std.

$D(\mathfrak{X}) := \Gamma(\mathfrak{X}, D_{\mathfrak{X}})$. diff op on $\mathfrak{X}$.

$\tau : \mathfrak{g} \hookrightarrow D(\mathfrak{X})$.

$\tau_d : \mathfrak{g} \hookrightarrow D(\mathfrak{X})$. $x \longmapsto \tau(x) - d\,\mathrm{tr}(x)$.

Harish-Chandra isom.

Thm [Gan-Ginzburg 2006].

$$\forall d \in \mathbb{C}, \quad \left(D(\mathfrak{X}) / D(\mathfrak{X}) \tau_d(\mathfrak{g}) \right)^G \cong A_{d-1}.$$

$\rightsquigarrow$ Hamiltonian reduction functor.

$$\mathbb{H}_d : (D(\mathfrak{X}), \mathfrak{g})\text{-mod}^{f.g, ss} \longrightarrow A_{d-1}\text{-mod}^{f.g}.$$

$$\begin{array}{ccc} \cup\!\!\!\mid & & \cup\!\!\!\mid \\ M & \longmapsto & M^{\tau_d(\mathfrak{g})} \\ ? & & \mathcal{O}(A_{d-1}) \\ & & \cup \\ & & eL_{d-1}. \end{array}$$

Def: A regular f.g $\mathcal{D}(\mathcal{X})$-module $\mathcal{M}$ is mirabolic if

$$SS(\mathcal{M}) \subseteq \{ (x,y,i,j) \in \mathfrak{g} \times \mathcal{N} \times T^*V \mid [x,y]+ij=0 \}$$

$$\underbrace{\mathrm{Hilb}_{\mathcal{X}}}_{\sim \;\cup\!\mid} \xrightarrow{/G} \mathrm{Hilb}_{\mathcal{X}} \longrightarrow \mathfrak{h} \times \{0\} /\!/ W$$

$\sim$ Character sheaves.

mirabolic D-module.

$$\mathbb{H}_d: \quad \mathcal{C}(\mathcal{X}) / \ker \mathbb{H}_d \xrightarrow{\sim} \mathcal{O}(A_{d-1})$$

$F\mathcal{C}^G(\mathcal{X})$: G-equivariant $\underset{\sim}{\text{filtered}}$ (good filtration) mirabolic D-modules.

Descent functor.

$$\Phi: \quad F\mathcal{C}^G(\mathcal{X}) \longrightarrow \mathrm{Coh}(\mathrm{Hilb})$$

$$(\mathcal{M}, \underset{\sim}{F_{\mathcal{M}}}) \longmapsto \mathrm{Desc}\left(\mathrm{gr}_{F_{\mathcal{M}}} \mathcal{M} \Big|_{\widetilde{\mathrm{Hilb}}} \right)$$

$$
\begin{tikzcd}
FO(A_{-c-1}) \arrow[d, "\Omega"'] & & F\ell^G(\mathcal{X}) \arrow[ll, "IH_{-c}"'] \arrow[dll] \arrow[d, "\Phi"] \\
FO(A_c) \arrow[rr, "GS"] & & Coh(Hilb)
\end{tikzcd}
$$

Fourier transform.
$x \leftrightarrow y$.

$$
\begin{array}{ccc}
H_c & \overset{\frown}{\longleftrightarrow} & H_{-c} \\
\cup\!| & & \\
A_c & \longrightarrow & A_{-c-1}.
\end{array}
\qquad e.
$$

Thm [M.]. For $(M, F_M) \in F\ell^G(\mathcal{X})$ s.t.

- $M^G = M^{\mathcal{I}_{-c}(\mathfrak{g})}$.
- $F_M \sim$ order filtration.

$$\Phi(M, F_M) = GS \circ \Omega \circ IH_{-c}\,(M, F_M).$$

Rem: $M = D(\mathcal{X})/I$.

Cuspidal mirabolic D-module.

$s: \quad \mathcal{X} \longrightarrow \mathbb{C}.$

$$(x,v) \longmapsto \langle vol, \ v \wedge xv \wedge \cdots \wedge x^{n-1} v \rangle$$

e.g: $x = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$

$$s(x,v) = \det \begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{pmatrix}.$$

Vandermonte.

e.g. $x = \begin{pmatrix} 0 & x_1 & & * \\ & \ddots & \ddots & \\ 0 & & \ddots & x_{n-1} \\ & & & 0 \end{pmatrix}. \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$

$$s(x,v) = x_1 x_2^2 \cdots x_{n-1}^{n-1} v_n^n.$$

$$G \curvearrowright \mathcal{Y} = \{ (x v) \in \underbrace{\mathcal{N}}_{\text{nilpotent}} \times V \mid s(x,v) \neq 0 \}.$$

free, transitive

- $g \cdot s = \det(g) \cdot s$. $\quad s \sim \det$.

$\mathbb{C} \cdot s^{-c}$. local system on $\mathcal{Y}$.

Def: Cuspidal mirabolic D-module.

$\mathcal{N}_{-c}$ is the minimal extension of $\mathbb{C} \cdot s^{-c}$ to $\mathcal{X}$.

$\mathcal{N}_{-c} \sim$ cuspidal character sheaf by Lusztig.

Thm ([Calaque, Enriquez, Etingof '09]).

$$\Omega \, \mathbb{H}_{-c}(\mathcal{N}_{-c}) = eL_c$$

Goal: · G-equivariant structure

· Hodge filtration.

Restrict to $\mathfrak{b}$. $\quad \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

$$\mathcal{Y}_0 := \mathcal{Y} \cap (\mathfrak{n} \times V) = \{ s_0 := x_1 x_2^{2} \cdots x_{n-1}^{n-1} v_n \neq 0 \}.$$

$Z := (\mathfrak{n} \times V) \setminus \mathcal{Y}_0$. SNC divisor.

$$\mathbb{C}^* \hookrightarrow \mathbb{C}.$$

$$\mathcal{L}_0 := \mathbb{C} \cdot s_0^{-c}. \quad \text{local system on } \mathcal{Y}_0 \xrightarrow{i} \mathfrak{n} \times V.$$

B. $\mathcal{L} := i_{!*} \mathcal{L}_0 = \mathcal{D} \big/ \left(\sum \mathcal{D}(x_i \partial_{x_i} + ic) + \sum \mathcal{D} \partial_{v_j} \right)$

$\quad \mathcal{D} := \mathcal{D}(\mathfrak{n} \times V)$

$\rotatebox{-90}{$\supseteq$}$

$\mathcal{L}^{>-1}$ Deligne's canonical extension.

- locally free, extending $\mathcal{L}_0$.
- lattice $\mathcal{L}^{>-1} \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{n} \times V}(Z) = \mathcal{L}$
- eigenvalues of residues along $Z \subseteq (-1, 0]$ (log connection)

$B \supseteq L^{>-1} = \mathcal{O}_{n\times V}[S_0^{-c}]^{-1}$.

$$S_0^{-c} = x_1^{-c} x_2^{-2c} \cdots x_{n-1}^{-(n-1)c} V_n^{-m}.$$

$$[S_0^{-c}] = x_1^{-[c]} x_2^{-[2c]} \cdots x_{n-1}^{-[(n-1)c]} V_n^{-m}.$$

$$S_0^{-c}[S_0^{-c}]^{-1}. \quad \checkmark.$$

$$F^k L^{>-1} = L^{>-1} \cap i_* F^k L_0 = \begin{cases} L^{>-1} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

$$F^k L = \sum F^i_{ord} D\, F^{k-i} L^{>-1}$$

$$\mathrm{gr}_F L = \mathcal{O}_{SS(L)}[S_0^{-c}]^{-1}.$$

$$B \supseteq [S_0^{-c}]^{-1}.$$

$$\downarrow$$

$$B/[B,B] \simeq T.$$

$$-\mu = -([c], [2c]-[c], \cdots, m-[(n-1)c])$$

Go to $\mathcal{N}_{-c}$.

- Induction of MHM. ([Archar]).
- Laumon's formula for D-module pushforward.
- Fourier transform for MHM ([Chen-Dirks., 2021])

$\mathbb{F}(\mathcal{N}_{-c}) \cong \mathcal{N}_{-c}$.

- $\operatorname{gr} \mathcal{N}_{-c}$ is CM [Saito].

$$\operatorname{gr} \mathcal{N}_{-c}\big|_{\widetilde{\mathrm{Hilb}}} = j_* \pi^* \mathcal{L}_u$$

$$\begin{tikzcd} \widetilde{\mathrm{Hilb}} & \\ {}_r\mathcal{Z}_r \arrow[u, hook, "j"] \arrow[r, "\pi"] & G/B \end{tikzcd}$$

$$\begin{tikzcd} & \boxed{G/B \times \mathfrak{X}} \arrow[dl] \arrow[dr] & \\ \mathfrak{X} & & \mathfrak{X} \end{tikzcd}$$

L lives on $\mathfrak{g} \times V$.

$\mathfrak{Z}$.

$\downarrow$

$\mathcal{X}$.

$$\begin{tikzcd} MHM_B(\mathcal{X}) \arrow[r, squiggly, "\mathrm{Ind}_B^G"] & MHM_G(G \times^B \mathcal{X}) \arrow[d, "\text{pushforward}"] \\ & MHM_G(\mathcal{X}) \end{tikzcd}$$

$$\begin{tikzcd} T^*(G \times^B \mathcal{X}) & G/B \times T^*\mathcal{X} \arrow[l, hook', "\rho"'] \arrow[r, "\varpi"] & T^*\mathcal{X} \\ & & \mathfrak{Z}_r \arrow[u, hook] \end{tikzcd}$$

$$\mathrm{gr}(p_+ M) = \varpi_*(L\rho^* \mathrm{gr} M \otimes \omega)$$

$$SS(M)\big|_{\mathfrak{Z}_r}.$$