

I: Some Homological algebra

\mathcal{A} - Abelian cat-y

$$\text{Ch}^b(\mathcal{A}) \xrightarrow{\text{mod homotopy}} K^b(\mathcal{A}) \xrightarrow{\text{invert quasi-iso}} D^b(\mathcal{A})$$

$$\cup \uparrow \\ K^b(\text{Proj } \mathcal{A})$$

fully faithful

sometimes an equivalence
(with finite proj-ve)
dimension

Even when $K^b(\text{Proj } \mathcal{A}) \cong D^b(\mathcal{A})$, the latter is better for some applications, eg. rt. der. functor.

Theme ① \hookrightarrow ② always fully faithful, sometimes eq., ② is sometimes better.

Next example: DG algebra

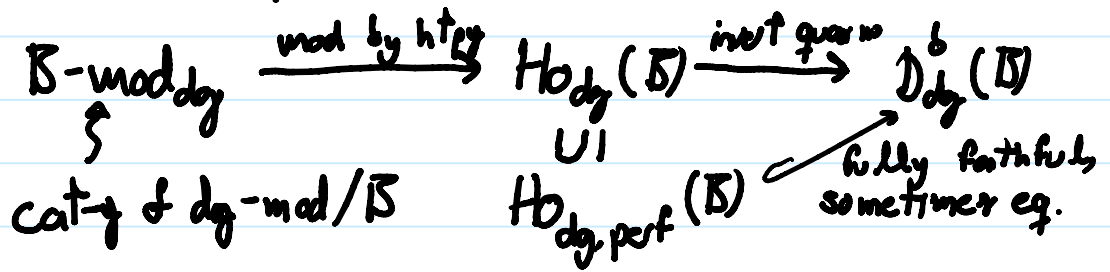
Let \mathcal{B} be a dg-dg/ k . This means

- ① \mathcal{B} is a chain cplx. of v.s. $\dots \rightarrow \mathcal{B}^{-1} \xrightarrow{d} \mathcal{B}^0 \xrightarrow{d} \mathcal{B}^1 \rightarrow \dots$
- ② Mult. map $m: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ that is a chain map, i.e. d obeys Leibniz rule
- ③ (\mathcal{B}, m) is an algebra

A dg-module M over \mathcal{B} consists of

- ① Chain cplx. M .
- ② Mult. map $m: \mathcal{B} \otimes M \rightarrow M$ that is a chain map.

- ② Mult. map $m: B \otimes M \rightarrow M$ that is a chain map.
- ③ Map from ② makes M into a B -module



Perfect = underlying module is free of finite rank/ B

$$Ho_{dg,perf}(B) \hookrightarrow D_{dg}^b(B) \text{ fits the theorem}$$

A ∞ -module M over a dg-dg B consists of:

- 1) Chain cplx. M .
- 2) Chain map $B \otimes M \rightarrow M$.
- 3) The maps $B \otimes B \otimes M \rightarrow M$ are homotopic by a specified homotopy

4) Choice of homotopy $B \otimes B \otimes B \otimes M$ is governed by specified "higher homotopy"

5) Keep going...

A ∞ -chain maps } Equiv coming from defn of "mg"/"module"
 A ∞ -homotopies } are only imposed up to homotopy

$$B\text{-mod}_{A\infty} \xrightarrow{\text{mod by hty}} Ho_{\infty}(B) \xrightarrow{\text{invert qisom}} D_{\infty}^b(B)$$

Miracle Under mild assumptions on B , qisom are already inv. up to homotopy and

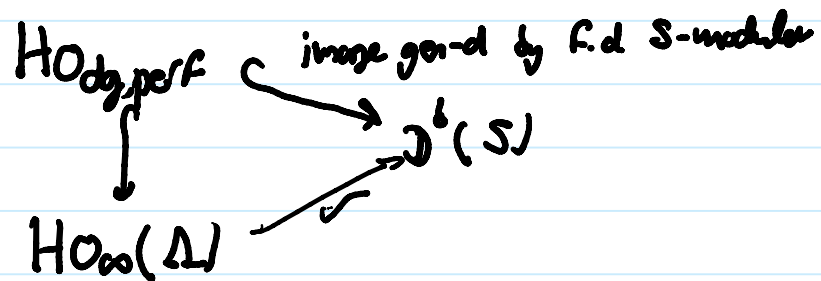
invariants under mild assumptions on \mathbb{A}^1 , \mathbb{A}^1 isos are already inv. up to homotopy and

$$D_{\infty}^{\delta}(\mathbb{B}) = D_{dg}^{\delta}(\mathbb{B}).$$

(Keller, Lefevre-Hasegawa)

$$Ho_{dg,perf}(\mathbb{B}) \hookrightarrow Ho_{\infty}(\mathbb{B}) \text{ fits the theorem.}$$

E.g. ① $\mathbb{B} = \Lambda^{\bullet} V$, $\deg V = 1$.
Classical Koszul duality, $D^{\delta}(\Lambda) = D^{\delta}(\text{Sym} V^{\bullet})$.



II. - Party sheaves, etc

Let \mathcal{P} be one of

1) (tors eq) party sh. on G/B or another variety

2) Schubert bimod

3) EW diagrammatic cat-y

} Any Coxeter group

We'll see why $K^{\delta} \mathcal{P}$ is important

$H = \text{Sym}(V)$. $\deg V = 2$. a var that acts on a module

$H = \text{Sym}(V)$, $\deg V = 2$, a ring that acts on graded Hom spaces in \mathcal{P}

e.g. could take

$$3) H = H_T(\mathcal{P}) = \text{Sym}(k^*)$$

$$2), 3) H = \text{Sym}(h^*) \otimes \text{Sym}(h) = \text{Sym}(h \oplus h^*)$$

or just one copy of $\text{Sym}(h^*)$.

Problem Replace $k^b \mathcal{P}$ by a new triang. cat-y \mathcal{P} which the action of H is trivial

Think: $k \otimes_{\mathbb{H}} (-)$

e.g.

- 1) Forget equivariance
- 2) Kill one copy of $\text{Sym}(h^*) \sim$ Serre module

3 Approaches to the problem

1) "Naive" Let $\bar{\mathcal{P}} = \text{cat-y}$ w/ same objects as \mathcal{P} , morphisms modified by $k \otimes_{\mathbb{H}} (-)$.

Work with $k^b \bar{\mathcal{P}}$.

Only works if Hom in \mathcal{P} were free/ H ("H-free situation")

2) "dog" Replace k by a dg-ctg g iso to k , flat/ H

2) "dg" Replace \mathbb{K} by a dg-alg g -iso to \mathbb{K} , flat/H
 eg let $\mathbb{B} = H \otimes_{\mathbb{Z}} \Delta V$ equipped w/ some differential
 (eg Koszul resol. of \mathbb{K}/H)

$$H_{\text{dg, perf}}(\mathbb{B}, \mathcal{P})$$

• Objects: Pairs (\mathcal{F}, θ)

\mathcal{F} - ordinary chain cplx of obj. in \mathcal{P} .

$\theta: \mathbb{B} \rightarrow \underline{\text{End}}(\mathcal{F})$ a dg-alg hom. st $\underline{\text{End}}(\mathcal{F})$ is free/ \mathbb{B}

3) "A ∞ " $H_{\infty}(\mathbb{B}, \mathcal{P})$

Objects: (\mathcal{F}, θ)

\mathcal{F} as before

θ an A_{∞} -alg hom (btw. dg-alg)

E.g 2) $\mathcal{P} = \mathbb{C}^n$ -eq. part. sh. on \mathbb{C}^n

$$K^b \mathcal{P} = D^b(\text{q.v.s.})$$

$H_{\text{dg, perf}}(\mathbb{B}, \mathcal{P})$ contains loc. systems w/ unip. monodromy

$H_{\infty}(\mathbb{B}, \mathcal{P})$ - contains the "pro unip. loc. system"

"Thm" ① $\text{Ho}_{\text{dg, perf}}(\mathcal{B}, \mathcal{P}) \hookrightarrow \text{Ho}_{\infty}(\mathcal{B}, \mathcal{P})$ fully faithful
 sometimes eq.

② In the H-free situation,

$$K^b \bar{\mathcal{P}} \cong \text{Ho}_{\text{dg, perf}}(\mathcal{B}, \mathcal{P}) \subseteq \text{Ho}_{\infty}(\mathcal{B}, \mathcal{P})$$

II. Application

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{i} & X & \xrightarrow{j} & X_n \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \xrightarrow{\quad} & \mathbb{C} & \xrightarrow{\quad} & \mathbb{C}^s
 \end{array}$$

Assume $\exists \mathbb{C}^r \hookrightarrow X$, part of sh on X_0 are in the
 H-free situation, $H = H_{\mathbb{C}^r}(\mathbb{C}^s)$

Classical formula $i^* j_* \exp_* \exp^* \mathcal{F}$

$$\text{Unip. part: } i^* j_* (\mathcal{F} \otimes \mathcal{F}^{-1} \mathcal{L}_{\infty}) [?]$$

\uparrow
 ∞ -rank par. unip.