Pramod Achar

quinta-feira, 20 de junho de 2019 14:02

I: Some Homological algebra

$\mathcal{A}$ - Abelian cat-y

$$\begin{tikzcd}
Ch^b(\mathcal{A}) \arrow[r, "\text{mod homotopy}"] & K^b(\mathcal{A}) \arrow[r, "\text{invert quasi-iso}"] & D^b(\mathcal{A}) \\
 & K^b(\text{Proj}\,\mathcal{A}) \arrow[u, phantom, "\cup" rotate=90] \arrow[ur, hook, "\text{fully faithful}"'] &
\end{tikzcd}$$

sometimes an equivalence (when finite proj-ve) dimension

Even when $K^b(\text{Proj}\,\mathcal{A}) \simeq D^b(\mathcal{A})$, the latter is better for some applications, eg rt. der. functors.

<u>Theme</u> ① $\hookrightarrow$ ② always fully faithful, sometimes eq., ② is sometimes better

<u>Next example</u>: DG algebra

Let B be a dg-alg/$\mathbb{k}$. This means

① B is a chain cplx. of v.s. $\to B^{-1} \xrightarrow{d} B^0 \xrightarrow{d} B^1 \to$
② Mult. map $m: B \otimes B \to B$ that is a chain map, ie d obeys Leibniz rule
③ (B, m) is an algebra

A dg-module M over B consists of
① Chain cplx. M.
② Mult. map $m: B \otimes M \to M$ that is a chain map.

① Chain cplx. M
② Mult. map $m : \mathbb{B} \otimes M \to M$ that is a chain map.
③ Map from ② makes M into a $\mathbb{B}$-module

$$\begin{array}{ccccc} \mathbb{B}\text{-mod}_{dg} & \xrightarrow{\text{mod by htpy}} & Ho_{dg}(\mathbb{B}) & \xrightarrow{\text{invert qisos}} & D^b_{dg}(\mathbb{B}) \\ \wr & & \cup\!\!\mid & \nearrow & \\ \text{cat of dg-mod}/\mathbb{B} & & Ho_{dg,perf}(\mathbb{B}) & & \end{array}$$

fully faithful, sometimes eq.

<u>Perfect</u> = underlying module is free of finite rank/$\mathbb{B}$

$Ho_{dg\,perf}(\mathbb{B}) \hookrightarrow D^b_{dg}(\mathbb{B})$ fits the theme

A_∞-module M over a dg-alg $\mathbb{B}$ consists of:

1) Chain cplx. M.
2) Chain map $\mathbb{B} \otimes M \to M$.
3) The maps $\mathbb{B} \otimes \mathbb{B} \otimes M \rightrightarrows M$ are homotopic by a specified homotopy

4) Choice of homotopy $\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B} \otimes M$ $\to M$ is governed by specified "higher homotopy"
5) Keep going...

A_∞-chain maps, A_∞-homotopies $\Big\}$ Eqns coming from defn of "ring"/"module" are only imposed up to homotopy

$$\mathbb{B}\text{-mod}_\infty \xrightarrow[A_\infty\text{-htpy}]{\text{mod by}} Ho_\infty(\mathbb{B}) \xrightarrow{\text{invert qisos}} D^b_\infty(\mathbb{B})$$

<u>Miracle</u> Under mild assumptions on $\mathbb{B}$, qisos are already inv. up to homotopy and

Miracle: Under mild assumption on B, q-isos are already inv. up to homotopy and

$$D^b_\infty(B) = D^b_{dg}(B).$$

(Keller, Lefevre-Hasegawa)

$Ho_{dg\text{-}perf}(B) \hookrightarrow Ho_\infty(B)$ fits the theme.

E.g. ① $B = \Lambda^\cdot V$, $\deg V = 1$.
Classical Koszul duality, $D^b(\Lambda) = D^b(\mathrm{Sym}\, V^*)$.

$$\begin{tikzcd}
Ho_{dg,perf} \arrow[r, hook, "\text{image gen-d by f.d } S\text{-modules}"] \arrow[d, hook] & D^b(S) \\
Ho_\infty(\Lambda) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

II - Parity sheaves, etc

Let $\mathcal{P}$ be one of

1) (torus eq) parity sh. on G/B or another variety
2) Soergel bimods
3) EW diagrammatic cat-y

} Any Coxeter group

We've seen why $K^b\mathcal{P}$ is important

$H = \mathrm{Sym}(V)$, $\deg V = 2$, a ring that acts on graded

$H = \mathrm{Sym}(V)$, $\deg V = 2$, a ring that acts on graded Hom spaces in $\mathcal{P}$

e.g could take

1) $H = H^{\bullet}_{T}(pt) = \mathrm{Sym}(\mathfrak{t}^*)$

2), 3) $H = \mathrm{Sym}(\mathfrak{h}^*) \otimes \mathrm{Sym}(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h} \oplus \mathfrak{h}^*)$
or just one copy of $\mathrm{Sym}(\mathfrak{h}^*)$.

Problem Replace $K^b\mathcal{P}$ by a new triang cat-y in which the action of H is trivial

Think: $\mathbb{K} \otimes_H (\ \ )$

e.g
1) Forget equivariance
2) Kill one copy of $\mathrm{Sym}(\mathfrak{h}^*) \rightsquigarrow$ Soergel modules

3 Approaches to the problem

1) "Naive" Let $\overline{\mathcal{P}}$ = cat-y w/ same objects as $\mathcal{P}$, morphisms modified by $\mathbb{K} \otimes_H (-)$.

Work with $K^b\overline{\mathcal{P}}$.

Only works if Hom in $\mathcal{P}$ were free/H ("H-free situation")

2) "dg" Replace $\mathbb{K}$ by a dg-alg q-iso to $\mathbb{K}$, flat/H

2) "dg" Replace $\Bbbk$ by a dg-alg q-iso to $\Bbbk$, flat/H

eg let
$$\mathcal{B} = H \otimes_{\Bbbk} \Lambda V \text{ equipped w/ some differential}$$

(eg Koszul resoln. of $\Bbbk / H$)

$$\mathrm{Ho}_{dg,perf}(\mathcal{B}, \mathcal{P})$$

· Objects: Pairs $(\mathcal{F}, \theta)$

$\mathcal{F}$ - ordinary chain cplx of obj. in $\mathcal{P}$.

$\theta : \mathcal{B} \to \underline{\mathrm{End}}(\mathcal{F})$ a dg-alg hom. at $\underline{\mathrm{End}}(\mathcal{F})$ in func/$\mathcal{B}$

3) "A_∞" $\mathrm{Ho}_\infty(\mathcal{B}, \mathcal{P})$

Objects: $(\mathcal{F}, \theta)$

$\mathcal{F}$ as before,

θ an A_∞-alg hom (btw. dg. alg)

E.g 2) $\mathcal{P} = \mathbb{C}^\times$-eq. parity sh. on $\mathbb{C}^\times$

$$K^b \mathcal{P} = D^b(\text{gr. v.s})$$

$\mathrm{Ho}_{dg,perf}(\mathcal{B}, \mathcal{P})$ contains loc. systems w/ unip. monodromy

$\mathrm{Ho}_\infty(\mathcal{B}, \mathcal{P})$ - contains the "pro unip. loc. system"

[illegible]

"Thm" ① $\mathrm{Ho}_{dg,perf}(\mathcal{B}, \mathcal{P}) \hookrightarrow \mathrm{Ho}_\infty(\mathcal{B}, \mathcal{P})$ fully faith.
sometimes eq.

② In the H-free situation,

$$K^b \tilde{\mathcal{P}} \simeq \mathrm{Ho}_{dg,perf}(\mathcal{B},\mathcal{P}) \simeq \mathrm{Ho}_\infty(\mathcal{B},\mathcal{P})$$

II. Application

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook', "j"'] \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{C} & \mathbb{C}^\times \arrow[l, hook'] \end{tikzcd}$$

<u>Assume</u> $\exists\ \mathbb{C}^\times \circlearrowright X$, parity sh on X_0 are in the H-free situation, $H = H_{\mathbb{C}^\times}(pt)$

<u>Classical formula</u> $i^* j_* \exp. \exp^* \mathcal{F}$

Unip. part: $i^* j_* (\mathcal{F} \otimes f^* \mathcal{L}_\infty)[?]$

∞-rank pro unip.