Informal algebraic geometry seminar
Exercise set 1

Recall that a degree d hypersurface in $\mathbb{P}^n$ is defined by the equation $f(x_0, \ldots, x_n) = 0$ where $[x_0 : \ldots : x_n]$ are homogeneous coordinates in $\mathbb{P}^n$ and f is a homogeneous polynomial of degree d. A *quadric* is a degree 2 hypersurface. A *line* in $\mathbb{P}^n$ through points $[a_0 : \ldots : a_n]$ and $[b_0 : \ldots : b_n]$ is the set of all points of the form

$$[\lambda a_0 + \mu b_0 : \ldots : \lambda a_n + \mu b_n], \quad [\lambda : \mu] \in \mathbb{P}^1.$$

1. Prove that a line in $\mathbb{P}^n$ either intersects a degree d hypersurface $\{f = 0\}$ in at most d points or is contained in it.

2. Find the dimension of the space of (a) degree d homogeneous polynomials in x_0, x_1, x_2 (b) degree d homogeneous polynomials in x_0, x_1, x_2, x_3.

3. Use Problem 2 to prove that (a) there is a unique quadric in $\mathbb{P}^2$ through 5 points in general position (b) there is a unique quadric in $\mathbb{P}^3$ through 9 points in general position.

4. The *Segre quadric* S in $\mathbb{P}^3$ is defined by the equation

$$\det \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = x_0 x_3 - x_1 x_2 = 0.$$

(a) Prove that S contains infinitely many lines (b) Find all lines contained in S (c)* Prove that S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

5. (a) Prove that any nondegenerate quadric in $\mathbb{P}^n$ can be tranformed to $\{x_0^2 + \ldots + x_n^2 = 0\}$ by some linear change of variables (here you need to work over $\mathbb{C}$) (b) Prove that any nondegenerate quadric in $\mathbb{P}^3$ can be tranformed to the Segre quadric by some linear change of variables (c) Use (b) and Problem 4 to conclude that any nondegenerate quadric in $\mathbb{P}^3$ contains infinitely many lines:

6. Given 4 lines $\ell_1, \ell_2, \ell_3, \ell_4$ in general position in $\mathbb{P}^3$, how many lines intersect them all? We will solve this problem using Schubert calculus in class, but here's a more elementary solution:

(a) Pick 3 points in general position in each of the lines ℓ_1, ℓ_2, ℓ_3, this is 9 points in total. By Problem 3 there is a quadric Q which passes through these 9 points. Prove that the lines ℓ_1, ℓ_2, ℓ_3 are contained in Q.

(b) Prove that any line intersecting ℓ_1, ℓ_2, ℓ_3 is also contained in Q

(c) Find all lines intersecting $\ell_1, \ell_2, \ell_3, \ell_4$ by intersecting ℓ_4 with Q.

7. Assume that n hypersurfaces $\{f_1 = 0\}, \ldots, \{f_n = 0\}$ defined by homogeneous polynomials of degrees $d_1, \ldots, d_n$ in $x_0, \ldots, x_n$ intersect transversally in $\mathbb{P}^n$. How many points of intersection are there?

8. Let X be a degree d hypersurface in $\mathbb{P}^n$, and let $\alpha \in H^2(X)$ be the class of a hyperplane section. Compute α^{n-1}.

9. Prove that the set of all lines in $\mathbb{P}^n$ is naturally isomorphic to the Grassmannian $Gr(2, n+1)$.

10. (a) Find the number of k-tuples of linearly independent vectors in n-dimensional space over a finite field with q elements.

(b) Find the number of invertible $k \times k$ matrices over a finite field with q elements.

(c) Find the number of points in the Grassmannian $Gr(k, n)$ over a finite field with q elements.