Informal algebraic geometry seminar
Exercise set 2

Schubert calculus

1. (a) Use Giambelli's formula to express Schubert cycle $\sigma_{3,2}$ in terms of σ_i.
(b) Use (a) and Pieri rule to compute $\sigma_{3,2} \cdot \sigma_{3,2}$.

2. We know from Giambelli's formula that classes $\sigma_1, \ldots, \sigma_{n-k}$ generate the cohomology ring of the Grassmannian $G(k, n)$. What are the relations in this ring?
(a) Use Giambelli's formula to express $\sigma_{1^s} = \sigma_{1,\ldots,1}$ in terms of σ_i
(b) Write the relations in σ_i expressing the fact that $\sigma_{1^s} = 0$ for $s = k+1, \ldots.n$
(c) In the case of $G(2, 4)$, we have two generators σ_1 and σ_2, and two relations in (b):

$$\sigma_{1,1,1} = \sigma_{1,1,1,1} = 0.$$

Use Giambelli's formula to write these relations explicitly, and verify that they define a 6-dimensional algebra

$$H(2,4) = \mathbb{Q}[\sigma_1, \sigma_2]/(\sigma_{1,1,1}, \sigma_{1,1,1,1}),\ \dim H(2,4) = 6.$$

(d) Conclude that $H(2, 4) = H^*(Gr(2, 4))$ and all other relations follow from these two.
(e)* Consider the algebra

$$H(k, n) = \mathbb{Q}[\sigma_1, \ldots, \sigma_{n-k}]/(\sigma_{1^{k+1}}, \ldots \sigma_{1^n}).$$

Prove that $\dim H(k, n) = \binom{n}{k}$ and conclude that $H(k, n) = H^*(Gr(k, n))$ for all k and n.

Plücker embedding

Recall that the Grassmannian $G(k, n)$ can be defined as the space of $k \times n$ matrices of rank k, modulo multiplication by a nondegenerate $k \times k$ matrix on the left.

3. For all k-element subsets $I \subset \{1, \ldots, n\}$ and a $k \times n$ matrix M define $\Delta_I(M)$ to be the minor of M in columns labeled by I.
(a) Prove that multiplication of M by a nondegenerate $k \times k$ matrix on the left multiplies all minors $\Delta_I(M)$ by the same number.
(b) Conclude that we have a map from $G(k, n)$ to the projective space of dimension $\binom{n}{k} - 1$ which sends a matrix M to a point with homogeneous coordinates $\Delta_I(M)$. It is callled the *Plücker embedding*, and Δ_I are called *Plücker coordinates*.
(c) Prove that the open chart $\{M : \Delta_I(M) \neq 0\}$ in $G(k, n)$ is isomorphic to the affine space of dimension $k(n - k)$.

4. In case of $G(2, 4)$ we have 6 Plücker coordinates which define a map $G(2, 4) \to \mathbb{P}^5$.
(a) Prove that they satisfy quadratic relation

$$\Delta_{1,2}\Delta_{3,4} - \Delta_{1,3}\Delta_{2,4} + \Delta_{1,4}\Delta_{2,3} = 0.$$

(b) Conclude that $G(2, 4)$ is isomorphic to a quadric in $\mathbb{P}^5$ via Plücker embedding.
(c) Write down explicit coordinates in charts $\{\Delta_I \neq 0\}$ in this case.

5. In case of $G(2, 5)$ we have 10 Plücker coordinates which define a map $G(2, 5) \to \mathbb{P}^9$.
(a) Use previous problem to prove that $G(2, 5)$ can be presented as the intersection of 5 quadrics in $\mathbb{P}^9$.
(b)* Observe that the dimension of $G(2, 5)$ equals $2 \times 3 = 6$, and hence $G(2, 5)$ is *not a complete intersection* in $\mathbb{P}^9$, that is, the number of defining equations does not match its codimension.