Tutorial on
Schubert Varieties and Schubert Calculus

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Philosophy

"Combinatorics is the equivalent of nanotechnology in mathematics."

Outline

1. Background and history of Grassmannians
2. Schur functions
3. Background on Flag Manifolds
4. Schubert polynomials
5. The Big Picture

Enumerative Geometry

Approximately 150 years ago... Grassmann, Schubert, Pieri, Giambelli, Severi, and others began the study of *enumerative geometry*.

Early questions:

- What is the dimension of the intersection between two general lines in $\mathbb{R}^{\mathbf{2}}$?
- How many lines intersect two given lines and a given point in $\mathbb{R}^{\mathbf{3}}$?
- How many lines intersect four given lines in $\mathbb{R}^{\mathbf{3}}$?

Modern questions:

- How many points are in the intersection of 2,3,4,... Schubert varieties in general position?

Schubert Varieties

A *Schubert variety* is a member of a family of projective varieties which is defined as the closure of some orbit under a group action in a homogeneous space G/H.

Typical properties:

- They are all Cohen-Macaulay, some are "mildly" singular.
- They have a nice torus action with isolated fixed points.
- This family of varieties and their fixed points are indexed by combinatorial objects; e.g. partitions, permutations, or Weyl group elements.

Schubert Varieties

"Honey, Where are my Schubert varieties?"

Typical contexts:

- *The Grassmannian Manifold*, $G(n,d) = GL_n/P$.
- The Flag Manifold: Gl_n/B.
- Symplectic and Orthogonal Homogeneous spaces: Sp_{2n}/B, O_n/P
- Homogeneous spaces for semisimple Lie Groups: G/P.
- *Affine Grassmannians*: $\mathcal{L}_G = G(\mathbb{C}[z, z^{-1}])/\widetilde{P}$.

More exotic forms: matrix Schubert varieties, Richardson varieties, spherical varieties, Hessenberg varieties, Goresky-MacPherson-Kottwitz spaces, positroids.

Why Study Schubert Varieties?

1. It can be useful to see points, lines, planes etc as families with certain properties.
2. Schubert varieties provide interesting examples for test cases and future research in algebraic geometry, combinatorics and number theory.
3. Applications in discrete geometry, computer graphics, computer vision, and economics.

The Grassmannian Varieties

Definition. Fix a vector space V over $\mathbb{C}$ (or $\mathbb{R}$, $\mathbb{Q}_p, \ldots$) with basis $B = \{e_1, \ldots, e_n\}$. The *Grassmannian variety*

$$G(k, n) = \{k\text{-dimensional subspaces of } V\}.$$

Question.

How can we impose the structure of a variety or a manifold on this set?

The Grassmannian Varieties

Answer. Relate $G(k, n)$ to the $k \times n$ matrices of rank k.

$$U = \text{span}\langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4 \rangle \in G(3, 4)$$

$$M_U = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix}$$

- $U \in G(k, n) \iff$ rows of M_U are independent vectors in $V \iff$ some $k \times k$ minor of M_U is NOT zero.

Plücker Coordinates

- Define $f_{j_1,j_2,\ldots,j_k}$ to be the homogeneous polynomial given by the determinant of the matrix

$$\begin{bmatrix} x_{1,j_1} & x_{1,j_2} & \ldots & x_{1,j_k} \\ x_{2,j_1} & x_{2,j_2} & \ldots & x_{2,j_k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{kj_1} & x_{kj_2} & \ldots & x_{kj_k} \end{bmatrix}$$

- $G(k,n)$ is an open set in the Zariski topology on $k \times n$ matrices defined as the union over all k-subsets of $\{1,2,\ldots,n\}$ of the complements of the varieties $V(f_{j_1,j_2,\ldots,j_k})$.

- $G(k,n)$ embeds in $\mathbb{P}^{\binom{n}{k}})$ by listing out the Plücker coordinates.

The Grassmannian Varieties

Canonical Form. Every subspace in $G(k,n)$ can be represented by a unique $k \times n$ matrix in row echelon form.

Example.

$$U = \text{span}\langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4\rangle \in G(3,4)$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2,\quad 2e_1 + e_3,\quad 7e_1 + e_4\rangle$$

Subspaces and Subsets

Example.

$$U = \mathrm{RowSpan}\begin{bmatrix} 5 & 9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 0 & 9 & 7 & 9 & 1 & 0 & 0 & 0 \\ 4 & 6 & 0 & 2 & 6 & 4 & 0 & 3 & 1 & 0 \end{bmatrix} \in G(3,10).$$

$$\mathrm{position}(U) = \{3, 7, 9\}$$

Definition.

If $U \in G(k,n)$ and M_U is the corresponding matrix in canonical form then the columns of the leading 1's of the rows of M_U determine a subset of size k in $\{1, 2, \ldots, n\} := [n]$. There are 0's to the right of each leading 1 and 0's above and below each leading 1. This k-subset determines the *position* of U with respect to the fixed basis.

The Schubert Cell C_{j} in $G(k,n)$

Defn. Let $\mathrm{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$. A *Schubert cell* is

$$C_{\mathrm{j}} = \{U \in G(k,n) \mid \mathrm{position}(U) = \{j_1, \ldots, j_k\}\}$$

Fact. $G(k,n) = \bigcup C_{\mathrm{j}}$ over all k-subsets of $[n]$.

Example. In $G(3,10)$,

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix} \right\}$$

- Observe, $\dim(C_{\{3,7,9\}}) = 2 + 5 + 6 = 13.$
- In general, $\dim(C_{\mathrm{j}}) = \sum j_i - i.$

Schubert Varieties in $G(k,n)$

Defn. Given $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$, the *Schubert variety* is

$$X_\lambda = \text{Closure of } C_\lambda \text{ under Zariski topology.}$$

Question. In $G(3,10)$, which minors vanish on $C_{\{3,7,9\}}$?

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix} \right\}$$

Answer. All minors f_{j_1,j_2,j_3} with $\left\{ \begin{array}{c} 4 \leq j_1 \leq 8 \\ \text{or } j_1 = 3 \text{ and } 8 \leq j_2 \leq 9 \\ \text{or } j_1 = 3, j_2 = 7 \text{ and } j_3 = 10 \end{array} \right\}$

In other words, the canonical form for any subspace in $X_{\mathbf{j}}$ has 0's to the right of column j_i in each row i.

k-Subsets and Partitions

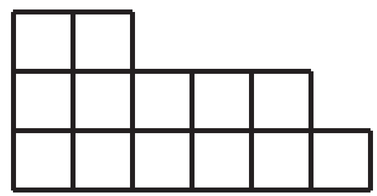
Defn. A *partition* of a number n is a weakly increasing sequence of non-negative integers

$$\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$$(2, 5, 6) \longrightarrow$$



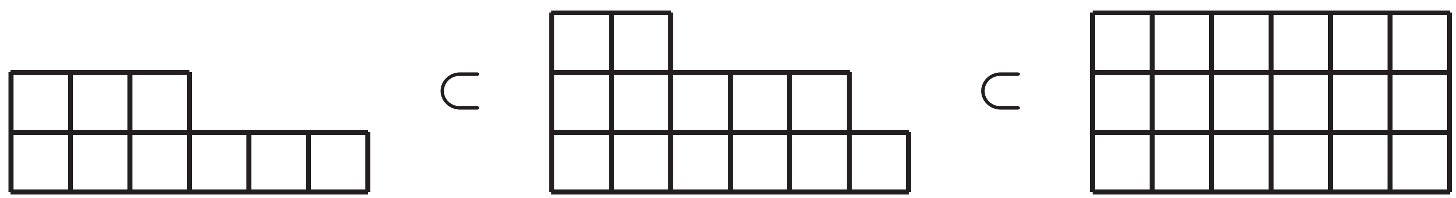
Fact. There is a bijection between k-subsets of $\{1, 2, \ldots, n\}$ and partitions whose Ferrers diagram is contained in the $k \times (n-k)$ rectangle given by

$$\mathbf{shape} : \{j_1 < \ldots < j_k\} \mapsto (j_1 - 1, j_2 - 2, \ldots, j_k - k).$$

A Poset on Partitions

Defn. A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

Defn. *Young's Lattice*
If $\boldsymbol{\lambda} = (\boldsymbol{\lambda_1} \leq \boldsymbol{\lambda_2} \leq \cdots \leq \boldsymbol{\lambda_k})$ and $\boldsymbol{\mu} = (\boldsymbol{\mu_1} \leq \boldsymbol{\mu_2} \leq \cdots \leq \boldsymbol{\mu_k})$ then $\boldsymbol{\lambda} \subset \boldsymbol{\mu}$ if the Ferrers diagram for $\boldsymbol{\lambda}$ fits inside the Ferrers diagram for $\boldsymbol{\mu}$.



Facts.

1. $\boldsymbol{X_{\mathrm{j}}} = \displaystyle\bigcup_{\mathbf{shape(i)} \subset \mathbf{shape(j)}} \boldsymbol{C_{\mathrm{i}}}.$

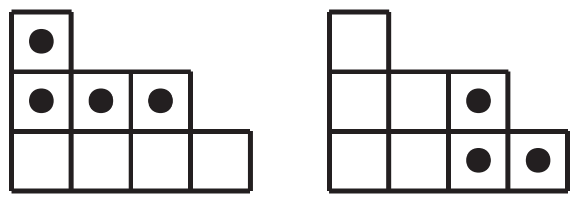
2. The dimension of $\boldsymbol{X_{\mathrm{j}}}$ is $|\mathbf{shape(j)}|$.

3. The Grassmannian $\boldsymbol{G(k,n)} = \boldsymbol{X_{\{n-k+1,\ldots,n-1,n\}}}$ is a Schubert variety!

Singularities in Schubert Varieties

Theorem. (Lakshmibai-Weyman) Given a partition λ. The singular locus of the Schubert variety X_λ in $G(k,n)$ is the union of Schubert varieties indexed by the set of all partitions $\mu \subset \lambda$ obtained by removing a hook from λ.

Example. $\mathbf{sing}((X_{(4,3,1)}) = X_{(4)} \cup X_{(2,2,1)}$



Corollary. X_λ is non-singular if and only if λ is a rectangle.

Enumerative Geometry Revisited

Question. How many lines intersect four given lines in $\mathbb{R}^3$?

Translation. Given a line in $\mathbb{R}^3$, the family of lines intersecting it can be interpreted in $G(2,4)$ as the Schubert variety

$$X_{\{2,4\}} = \overline{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}}$$

with respect to a suitably chosen basis determined by the line.

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Modern Solution. Use Schubert calculus!

Schubert Calculus/Intersection Theory

- Schubert varieties induce canonical basis elements of the cohomology ring $H^*(G(k,n))$ called *Schubert classes*: $[X_{\mathbf{j}}]$.

- Multiplication in $H^*(G(k,n))$ is determined by intersecting Schubert varieties with respect to generically chosen bases

$$[X_{\mathbf{i}}][X_{\mathbf{j}}] = \left[X_{\mathbf{i}}(B^1) \cap X_{\mathbf{j}}(B^2)\right]$$

- The entire multiplication table is determined by

Giambelli Formula: $[X_{\mathbf{i}}] = \det\left(e_{\lambda'_i - i + j}\right)_{1 \le i,j \le k}$

Pieri Formula: $$[X_{\mathbf{i}}]\, e_r = \sum [X_{\mathbf{j}}]$$

Intersection Theory/Schubert Calculus

- Schubert varieties induce canonical basis elements of the cohomology ring $H^*(G(k,n))$ called *Schubert classes*: $[X_{\mathbf{j}}]$.

- Multiplication in $H^*(G(k,n))$ is determined by intersecting Schubert varieties with respect to generically chosen bases

$$[X_{\mathbf{i}}][X_{\mathbf{j}}] = \left[X_{\mathbf{i}}(B^{\mathbf{1}}) \cap X_{\mathbf{j}}(B^{\mathbf{2}})\right]$$

- The entire multiplication table is determined by

  Giambelli Formula: $[X_{\mathbf{i}}] = \det \left(e_{\lambda'_i - i + j}\right)_{1 \le i,j \le k}$

  Pieri Formula: $[X_{\mathbf{i}}]\, e_r = \sum [X_{\mathbf{j}}]$

  where the sum is over classes indexed by shapes obtained from $\mathbf{shape(i)}$ by removing a vertical strip of r cells.

- $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ is the conjugate of the box complement of $\mathbf{shape(i)}$.

- e_r is the special Schubert class associated to $k \times n$ minus r boxes along the right col. e_r is a Chern class in the Chern roots $x_1, \ldots, x_n$.

Intersection Theory/Schubert Calculus

Schur functions S_λ are a fascinating family of symmetric functions indexed by partitions which appear in many areas of math, physics, theoretical computer science, quantum computing and economics.

- The Schur functions S_λ are symmetric functions that also satisfy

  Giambelli/Jacobi-Trudi Formula: $S_\lambda = \det\left(e_{\lambda'_i - i + j}\right)_{1 \leq i,j \leq k}$

  Pieri Formula: $$S_\lambda \, e_r = \sum S_\mu .$$

- Thus, as rings $H^*(G(k,n)) \approx \mathbb{C}[x_1, \ldots, x_n]^{S_n} / \langle S_\lambda : \lambda \not\subset k \times n \rangle$.

- Expanding the product of two Schur functions into the basis of Schur functions can be done via linear algebra:

$$S_\lambda S_\mu = \sum c^\nu_{\lambda,\mu} S_\nu .$$

- The coefficients $c^\nu_{\lambda,\mu}$ are non-negative integers called the Littlewood-Richardson coefficients.

Schur Functions

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an alphabet of indeterminants.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ and $\lambda_p = 0$ for $p \geq k$.

Defn. The following are equivalent definitions for the Schur functions $S_\lambda(X)$:

1. $S_\lambda = \det\left(e_{\lambda'_i - i + j}\right) = \det\left(h_{\lambda_i - i + j}\right)$
2. $S_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$ with indices $1 \leq i, j \leq m$.
3. $S_\lambda = \sum x^T$ summed over all *column strict tableaux* T *of shape* λ.
4. $S_\lambda = \sum F_{D(T)}(X)$ summed over all *standard tableaux* T *of shape* λ.

Schur Functions

Defn. $S_\lambda = \sum F_{D(T)}(X)$ over all *standard tableaux* T *of shape* λ.

Defn. A standard tableau T of shape λ is a saturated chain in Young's lattice from $\emptyset$ to λ. The descent set of T is the set of indices i such that $i+1$ appears northwest of i.

Example.

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 \\ \cline{1-3} 4 & 5 & 9 \\ \hline 1 & 2 & 3 & 6 & 8 \\ \hline \end{array} \qquad D(T) = \{3, 6, 8\}.$$

Defn. The fundamental quasisymmetric function

$$F_{D(T)}(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \ldots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \in D(T)$.

Littlewood-Richardson Rules

Recall if $S_\lambda S_\mu = \sum c^\nu_{\lambda,\mu} S_\nu$, then the coefficients $c^\nu_{\lambda,\mu}$ are *non-negative integers* called Littlewood-Richardson coefficients.

Littlewood-Richardson Rules.

1. Schützenberger: Fix a standard tableau T of shape ν. Then $c^\nu_{\lambda,\mu}$ equals the number of pairs of standard tableaux of shapes λ, μ which straighten under the rules of jeu de taquin into T.

2. Yamanouchi Words: $c^\nu_{\lambda,\mu}$ equals the number of column strict fillings of the skew shape ν/μ with λ_1 1's, λ_2 2's, etc such that the reverse reading word always has more 1's than 2's, more 2's than 3's, etc.

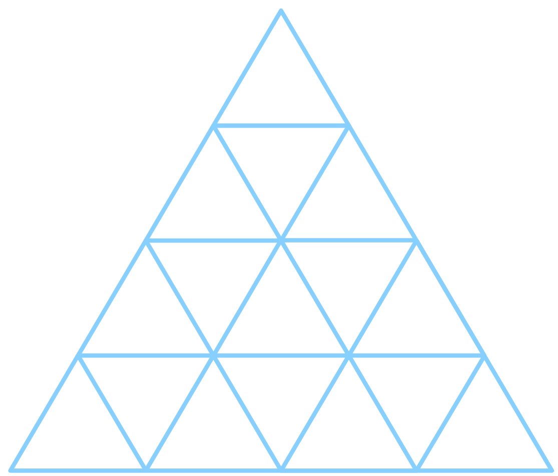
3. Remmel-Whitney rule: $c^\nu_{\lambda,\mu}$ equals the number of leaves of shape ν in the tree of standard tableaux with root given by the standard labeling of λ and growing on at each level respecting two adjacency rules.

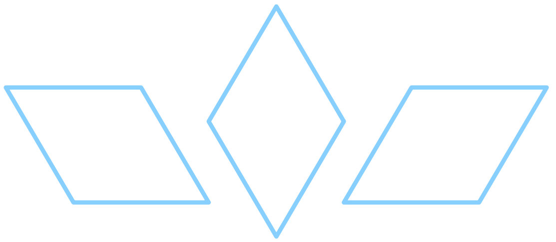
4. Knutson-Tao Puzzles: $c^\nu_{\lambda,\mu}$ equals the number of λ, μ, ν - puzzles.

5. Vakil Degenerations: $c^\nu_{\lambda,\mu}$ equals the number of leaves in the λ, μ-tree of checkerboards with type ν.

Knutson-Tao Puzzles

Example. (Warning: picture is not accurate without description.)





Vakil Degenerations

Show picture.

Enumerative Solution

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Enumerative Solution

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Solution.

$$\left[X_{\{2,4\}}\right] = S_{(1)} = x_1 + x_2 + \dots$$

By the recipe, compute

$$\left[X_{\{2,4\}}(B^1) \cap X_{\{2,4\}}(B^2) \cap X_{\{2,4\}}(B^3) \cap X_{\{2,4\}}(B^4)\right]$$

$$= S_{(1)}^4 = 2S_{(2,2)} + S_{(3,1)} + S_{(2,1,1)}.$$

Answer. The coefficient of $S_{2,2} = [X_{1,2}]$ is *2* representing the two lines meeting 4 given lines in general position.

Recap

1. $G(k,n)$ is the Grassmannian variety of k-dim subspaces in $\mathbb{R}^n$.

2. The Schubert varieties in $G(k,n)$ are nice projective varieties indexed by k-subsets of $[n]$ or equivalently by partitions in the $k \times (n-k)$ rectangle.

3. Geometrical information about a Schubert variety can be determined by the combinatorics of partitions.

4. Schubert Calculus (intersection theory applied to Schubert varieties and associated algorithms for Schur functions) can be used to solve problems in enumerative geometry.

Current Research

1. (Gelfand-Goresky-MacPherson-Serganova) Matroid stratification of $G(k,n)$: specify the complete list of Plücker coordinates which are non-zero. What is the cohomology class of the closure of each strata?

2. (Kodama-Williams, Telaska-Williams) Deodhar stratification using Go-diagrams. What is the cohomology class of the closure of each strata?

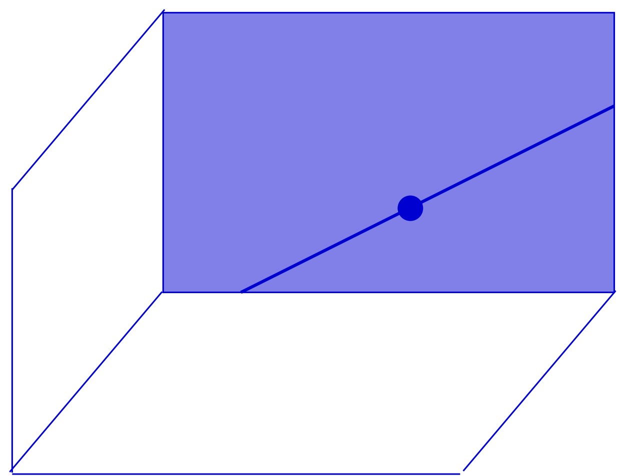
3. (MacPherson) What is a good way to triangulate Gr(k,n)?

The Flag Manifold

Defn. A *complete flag* $F_\bullet = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \leq i \leq n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \ldots f_n \rangle$ where $F_i = \operatorname{span}\langle f_1, \ldots, f_i \rangle$.

Example.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$



The Flag Manifold

Canonical Form.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4, \quad e_1 \rangle$$

$\mathcal{F}l_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n \subset \prod_{k=1}^n G(n,k)$

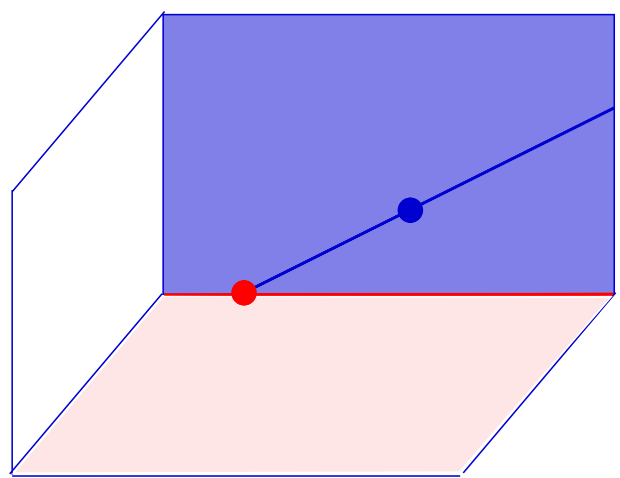
$= \{$complete flags $F_\bullet\}$

$= B \setminus GL_n(\mathbb{C}), \quad B =$ lower triangular mats.

Flags and Permutations

Example. $F_\bullet = \langle 2e_1 + e_2,\ \ 2e_1 + e_3,\ \ 7e_1 + e_4,\ \ e_1 \rangle \approx \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.



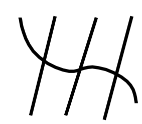
Many ways to represent a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = 2341 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

matrix notation | two-line notation | one-line notation | rank table

$$\begin{array}{cccc} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} =$$

1234



2341

$$= (1, 2, 3) = \#9$$

diagram of a permutation | string diagram | reduced word | position in lex order

The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

Defn. $C_w(E_\bullet) =$ All flags $F_\bullet$ with $\mathbf{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \mathrm{rk}(w[i,j])\}$$

Example. $F_\bullet = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$

Easy Observations.

- $\mathtt{dim}_{\mathbb{C}}(C_w) = l(w) = \#$ inversions of w.
- $C_w = w \cdot B$ is a B-orbit using the right B action, e.g.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{1,1} & 0 & 0 & 0 \end{bmatrix}$$

The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

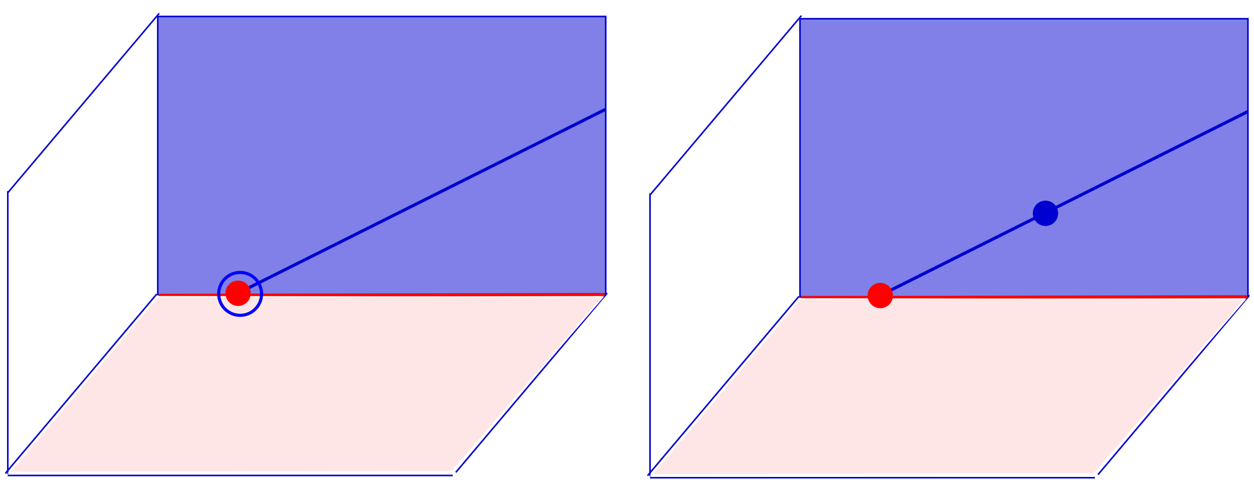
Defn. $X_w(E_\bullet)$ = Closure of $C_w(E_\bullet)$ under the Zariski topology

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.

Example. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}}$$

Why?.



Five Fun Facts

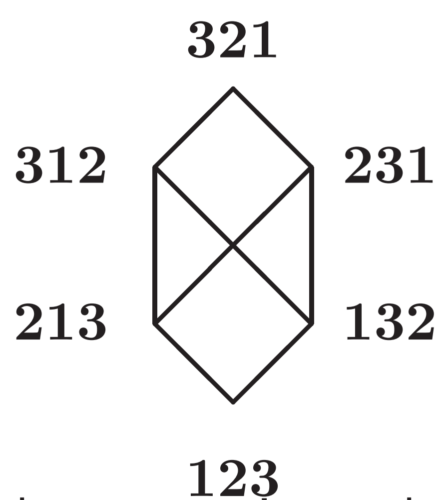
Fact 1. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \le w} C_v \qquad = \bigcup_{v \le w} X_v$$

Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

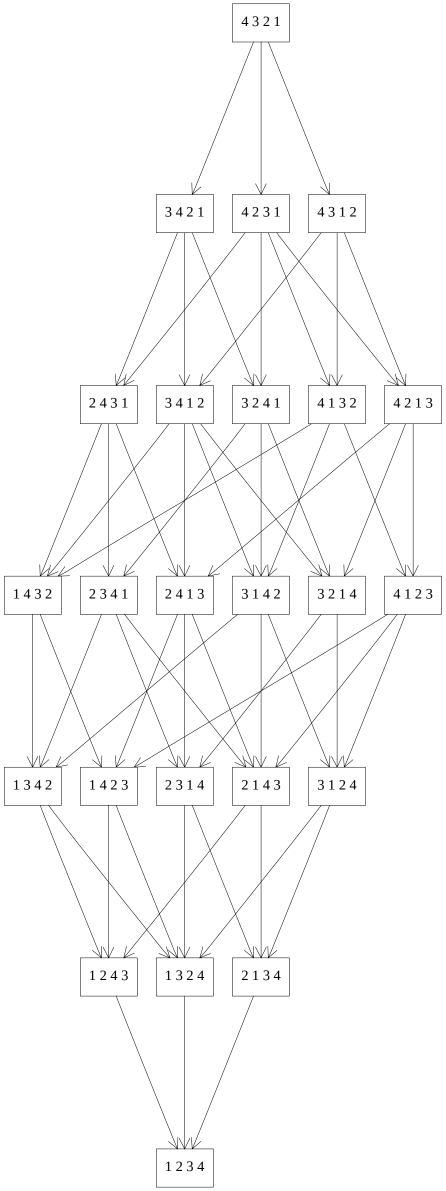
$$w < wt_{ij} \iff w(i) < w(j).$$

Example. Bruhat order on permutations in S_3.

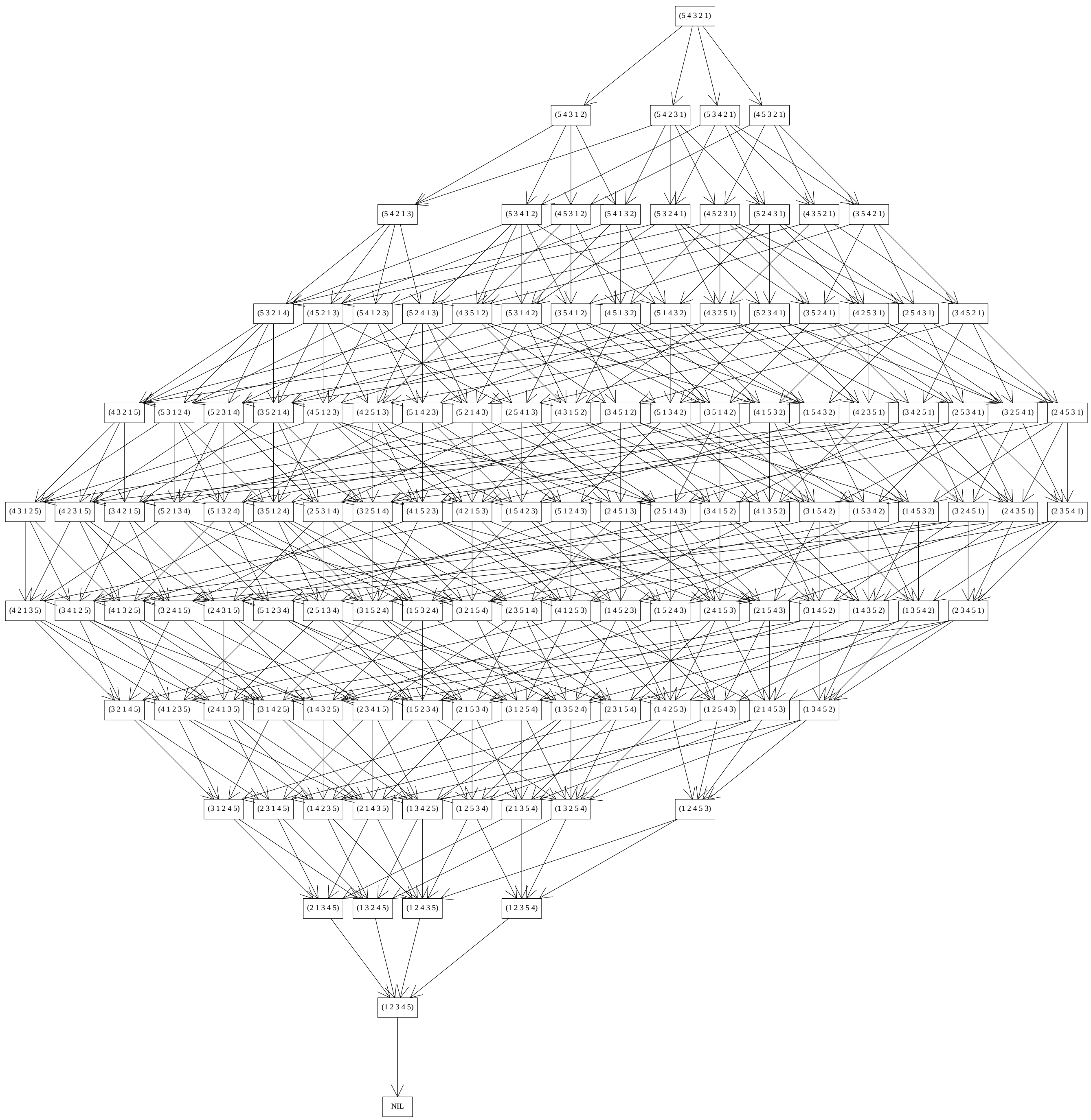


Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_4



Bruhat order on S_5



10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \iff X(w)$ rationally smooth
7. Order Complex of (u, v) is shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

Singularities in Schubert Varieties

Defn. X_w is *singular* at a point $p \iff$
$\mathbf{dim} X_w = l(w) <$ dimension of the tangent space to X_w at p.

Observation 1. Every point on a Schubert cell C_v in X_w looks locally the same. Therefore, $p \in C_v$ is a singular point $\iff$ the permutation matrix v is a singular point of X_w.

Observation 2. The singular set of a varieties is a closed set in the Zariski topology. Therefore, if v is a singular point in X_w then every point in X_v is singular. The irreducible components of the singular locus of X_w is a union of Schubert varieties:

$$\mathbf{Sing}(X_w) = \bigcup_{v \in \mathbf{maxsing(w)}} X_v.$$

Singularities in Schubert Varieties

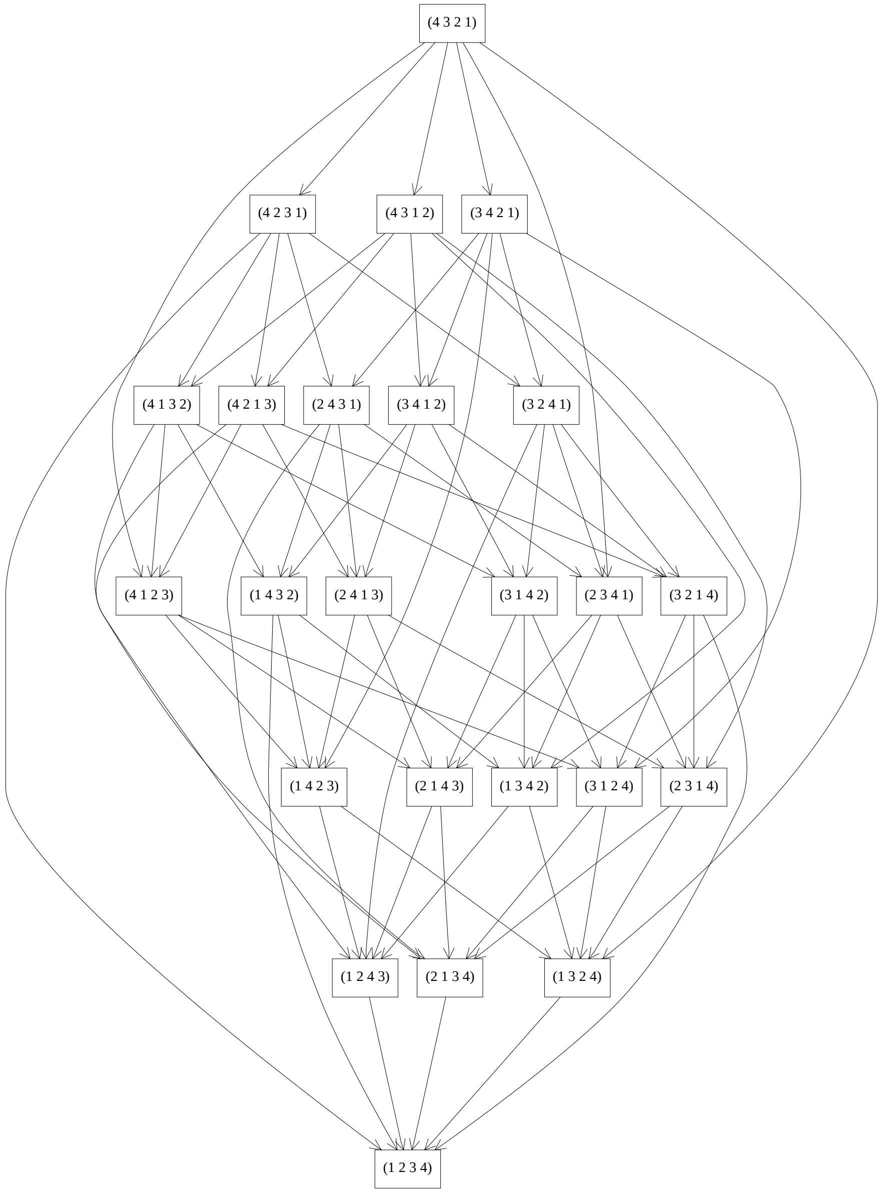
Fact 2. (Lakshmibai-Seshadri) A basis for the tangent space to X_w at v is indexed by the transpositions t_{ij} such that

$$vt_{ij} \leq w.$$

Definitions.

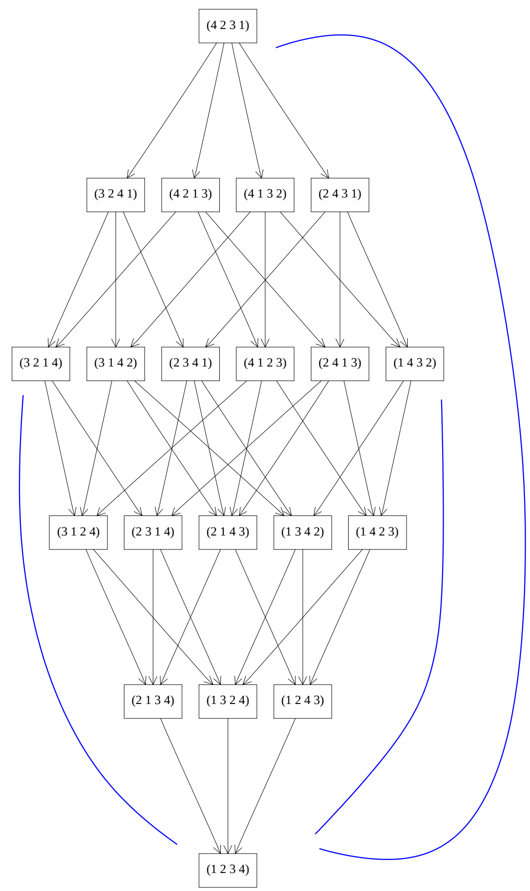
- Let T = invertible diagonal matrices. The T-fixed points in X_w are the permutation matrices indexed by $v \leq w$.
- If $v,\ vt_{ij}$ are permutations in X_w they are connected by a T-stable curve. The set of all T-stable curves in X_w are represented by the *Bruhat graph* on $[id, w]$.

Bruhat Graph in S_4



Tangent space of a Schubert Variety

Example. $T_{1234}(X_{4231}) = \text{span}\{x_{i,j} \mid t_{ij} \leq w\}$.



dim$X(4231)$=5 dim$T_{id}(4231) = 6 \implies X(4231)$ is singular!

Five Fun Facts

Fact 3. There exists a simple criterion for characterizing singular Schubert varieties using pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)
X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

$w = \mathbf{625431}$ contains $\mathbf{6241} \sim \mathbf{4231}$ $\implies$ X_{625431} is singular

Example: $w = \mathbf{612543}$ avoids $\mathbf{4231}$ $\&\mathbf{3412}$ $\implies$ X_{612543} is non-singul

Five Fun Facts

Fact 4. There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Theorem: Woo-Yong (Sept. 2004)

X_w is Gorenstein $\iff$

- w avoids $\mathbf{31542}$ and $\mathbf{24153}$ *with Bruhat restrictions* $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$

- for each descent d in w, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

See "A Unification Of Permutation Patterns Related To Schubert Varieties" by Henning Úlfarsson (arxiv 2012).

Five Fun Facts

Fact 5. Schubert varieties are useful for studying the cohomology ring of the flag manifold.

Theorem (Borel): $H^*(\mathcal{F}l_n) \cong \dfrac{\mathbb{Z}[x_1, \ldots, x_n]}{\langle e_1, \ldots e_n \rangle}$.

- The symmetric function $e_i = \displaystyle\sum_{1 \leq k_1 < \cdots < k_i \leq n} x_{k_1} x_{k_2} \ldots x_{k_i}$.
- $\{[X_w] \mid w \in S_n\}$ form a basis for $H^*(\mathcal{F}l_n)$ over $\mathbb{Z}$.

Question. What is the product of two basis elements?

$$[X_u] \cdot [X_v] = \sum [X_w] c_{uv}^w.$$

Cup Product in $H^*(\mathcal{F}l_n)$

One Answer. Use Schubert polynomials! Due to Lascoux-Schützenberger, Bernstein-Gelfand-Gelfand, Demazure.

- BGG: Set $[X_{id}] \equiv \prod_{i>j}(x_i - x_j) \in \frac{\mathbb{Z}[x_1,\ldots,x_n]}{\langle e_1,\ldots e_n\rangle}$

  If $\mathfrak{S}_w \equiv [X_w] \mathrm{mod} \langle e_1, \ldots e_n\rangle$ then

  $$\partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}} \equiv [X_{ws_i}] \text{ if } l(w) < l(ws_i)$$

- LS: Choosing $[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ works best because product expansion can be done without regard to the ideal!

- Here $\mathbf{deg}[X_w] = \mathbf{codim}(X_w)$.

Schubert polynomials for S_4

$$
\begin{aligned}
\mathfrak{S}_{w_0(1234)} &= 1 \\
\mathfrak{S}_{w_0(2134)} &= x_1 \\
\mathfrak{S}_{w_0(1324)} &= x_2 + x_1 \\
\mathfrak{S}_{w_0(3124)} &= x_1^2 \\
\mathfrak{S}_{w_0(2314)} &= x_1 x_2 \\
\mathfrak{S}_{w_0(3214)} &= x_1^2 x_2 \\
\mathfrak{S}_{w_0(1243)} &= x_3 + x_2 + x_1 \\
\mathfrak{S}_{w_0(2143)} &= x_1 x_3 + x_1 x_2 + x_1^2 \\
\mathfrak{S}_{w_0(1423)} &= x_2^2 + x_1 x_2 + x_1^2 \\
\mathfrak{S}_{w_0(4123)} &= x_1^3 \\
\mathfrak{S}_{w_0(2413)} &= x_1 x_2^2 + x_1^2 x_2 \\
\mathfrak{S}_{w_0(4213)} &= x_1^3 x_2 \\
\mathfrak{S}_{w_0(1342)} &= x_2 x_3 + x_1 x_3 + x_1 x_2 \\
\mathfrak{S}_{w_0(3142)} &= x_1^2 x_3 + x_1^2 x_2 \\
\mathfrak{S}_{w_0(1432)} &= x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2 \\
\mathfrak{S}_{w_0(4132)} &= x_1^3 x_3 + x_1^3 x_2 \\
\mathfrak{S}_{w_0(3412)} &= x_1^2 x_2^2 \\
\mathfrak{S}_{w_0(4312)} &= x_1^3 x_2^2 \\
\mathfrak{S}_{w_0(2341)} &= x_1 x_2 x_3 \\
\mathfrak{S}_{w_0(3241)} &= x_1^2 x_2 x_3 \\
\mathfrak{S}_{w_0(2431)} &= x_1 x_2^2 x_3 + x_1^2 x_2 x_3
\end{aligned}
$$

Cup Product in $H^*(\mathcal{F}l_n)$

Key Feature. Schubert polynomials are a positive sum of monomials and have distinct leading terms, therefore expanding any polynomial in the basis of Schubert polynomials can be done by linear algebra just like Schur functions.

Buch: Fastest approach to multiplying Schubert polynomials uses Lascoux and Schützenberger's transition equations. Works up to about $n = 15$.

Draw Back. Schubert polynomials don't prove c_{uv}^w's are nonnegative (except in special cases).

Cup Product in $H^*(\mathcal{F}l_n)$

Another Answer.

- By intersection theory: $[X_u] \cdot [X_v] = [X_u(E_\bullet) \cap X_v(F_\bullet)]$

- Perfect pairing: $[X_u(E_\bullet)] \cdot [X_v(F_\bullet)] \cdot [X_{w_0 w}(G_\bullet)] = c_{uv}^w [X_{id}]$

$$||$$

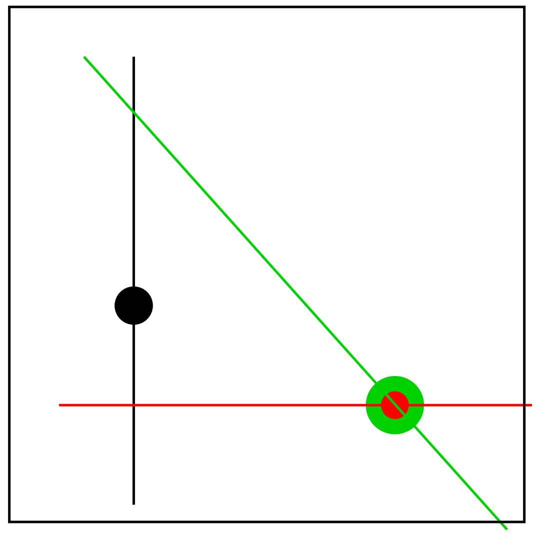
$$[X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)]$$

- The Schubert variety X_{id} is a single point in $\mathcal{F}l_n$.

Intersection Numbers: $c_{uv}^w = \# X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)$
Assuming all flags $E_\bullet$, $F_\bullet$, $G_\bullet$ are in sufficiently general position.

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:

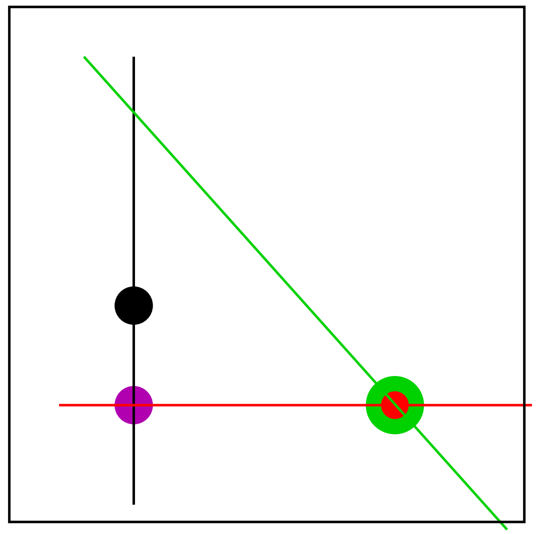


Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

$$
\begin{array}{c|ccc|}
 & R_1 & R_2 & R_3 \\
\hline
P_1 & & 1 & \\
P_2 & & & 1 \\
P_3 & 1 & & \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|}
G_1 & G_2 & G_3 \\
\hline
 & & 1 \\
 & 1 & \\
1 & & \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|}
B_1 & B_2 & B_3 \\
\hline
 & 1 & \\
1 & & \\
 & & 1 \\
\hline
\end{array}
$$

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:

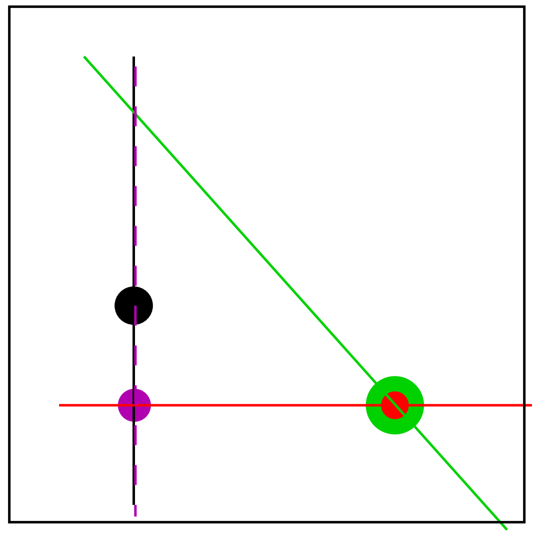


Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

$$
\begin{array}{c|ccc|c|ccc|c|ccc|}
 & R_1 & R_2 & R_3 & & G_1 & G_2 & G_3 & & B_1 & B_2 & B_3 \\
\hline
P_1 & & 1 & & & & & 1 & & & 1 & \\
P_2 & & & 1 & & & 1 & & & 1 & & \\
P_3 & 1 & & & & 1 & & & & & & 1 \\
\hline
\end{array}
$$

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:



Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

$$
\begin{array}{c|c|c|c|}
 & R_1 & R_2 & R_3 \\
\hline
P_1 & & 1 & \\
\hline
P_2 & & & 1 \\
\hline
P_3 & 1 & & \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|}
G_1 & G_2 & G_3 \\
\hline
 & & 1 \\
\hline
 & 1 & \\
\hline
1 & & \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|}
B_1 & B_2 & B_3 \\
\hline
 & 1 & \\
\hline
1 & & \\
\hline
 & & 1 \\
\hline
\end{array}
$$

Intersecting Schubert Varieties

Schubert's Problem. How many points are there usually in the intersection of d Schubert varieties if the intersection is 0-dimensional?

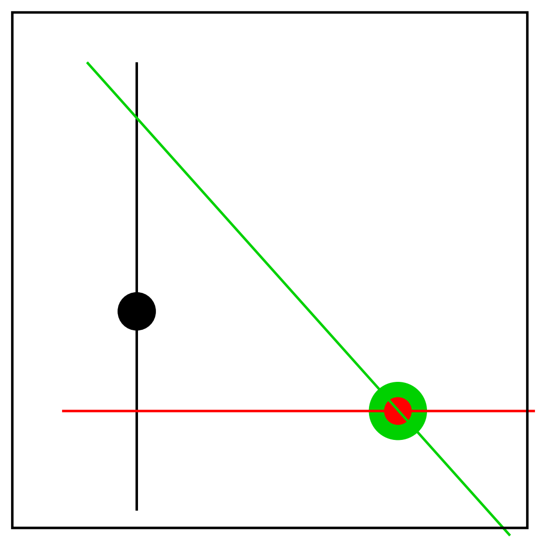
- Solving approx. n^d equations with $\binom{n}{2}$ variables is challenging!

Observation. We need more information on spans and intersections of flag components, e.g. $\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d})$.

Permutation Arrays

Theorem. (Eriksson-Linusson, 2000) For every set of d flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$, there exists a unique permutation array $P \subset [n]^d$ such that

$$\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d}) = \mathrm{rk}P[x].$$



	R_1	R_2	R_3
B_1			
B_2			
B_3	1	1	1

G_1

	R_1	R_2	R_3
B_1			
B_2			1
B_3	1	1	2

G_2

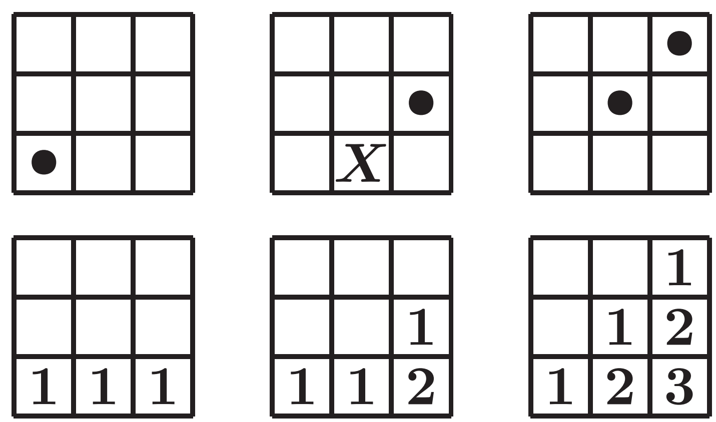
	R_1	R_2	R_3
B_1			1
B_2		1	2
B_3	1	2	3

G_3

Totally Rankable Arrays

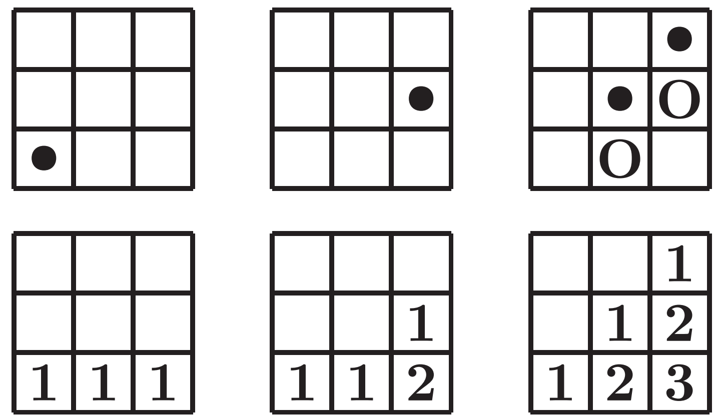
Defn. For $P \subset [n]^d$,

- $\mathrm{rk}_j P = \#\{k \mid \exists x \in P \text{ s.t. } x_j = k\}$.
- P is *rankable* of rank r if $\mathrm{rk}_j(P) = r$ for all $1 \leq j \leq d$.
- $y = (y_1, \ldots, y_d) \preceq x = (x_1, \ldots, x_d)$ if $y_i \leq x_i$ for each i.
- $P[x] = \{y \in P \mid y \preceq x\}$
- P is *totally rankable* if $P[x]$ is rankable for all $x \in [n]^d$.



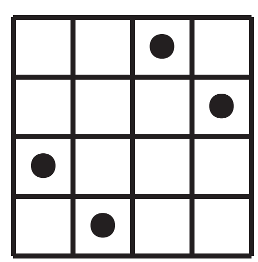
- Union of dots is totally rankable. Including X it is not.

Permutation Arrays



- Points labeled O are redundant, i.e. including them gives another totally rankable array with same rank table.

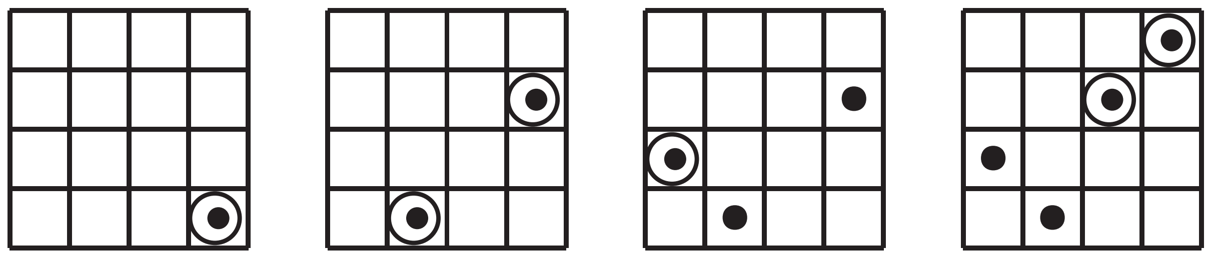
Defn. $P \subset [n]^d$ is a *permutation array* if it is totally rankable and has no redundant dots.

 $\in [4]^2$.

Open. Count the number of permutation arrays in $[n]^k$.

Permutation Arrays

Theorem. (Eriksson-Linusson) Every permutation array in $[n]^{d+1}$ can be obtained from a unique permutation array in $[n]^d$ by identifying a sequence of antichains.



This produces the 3-dimensional array

$$P = \{(4,4,1),(2,4,2),(4,2,2),(3,1,3),(1,4,4),(2,3,4)\}.$$

			4
		4	2
3			
	2		1

Unique Permutation Array Theorem

Theorem. (Billey-Vakil, 2005) If

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

is nonempty 0-dimensional intersection of d Schubert varieties with respect to flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$ in general position, then there exists a unique permutation array $P \in [n]^{d+1}$ such that

$$X = \{F_\bullet \mid \dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}}) = \mathrm{rk}P[x].\} \qquad (1)$$

Furthermore, we can recursively solve a family of equations for X using P.

Current Research

Open Problem. Can one find a finite set of rules for moving dots in a 3-d permutation array which determines the c_{uv}^{w}'s analogous to one of the many Littlewood-Richardson rules?

Recent Progress/Open question. Izzet Coskun's Mondrian tableaux. Can his algorithm be formulated succinctly enough to program without solving equations?

Open Problem. Give a minimal list of relations for $H^*(X_w)$. (See recent work of Reiner-Woo-Yong.)

Generalizations of Schubert Calculus for G/B

1993-2013: A Highly Productive Score.

$$\left\{ \begin{array}{l} \text{A: } GL_n \\ \text{B: } SO_{2n+1} \\ \text{C: } SP_{2n} \\ \text{D: } SO_{2n} \\ \text{Semisimple Lie Groups} \\ \text{Kac-Moody Groups} \\ \text{GKM Spaces} \end{array} \right\} \times \left\{ \begin{array}{l} \text{cohomology} \\ \text{quantum} \\ \text{equivariant} \\ \text{K-theory} \\ \text{eq. K-theory} \end{array} \right\}$$

Recent Contributions from: Bergeron, Berenstein, Billey, Brion, Buch, Carrell, Ciocan-Fontainine, Coskun, Duan, Fomin, Fulton, Gelfand, Goldin, Graham, Griffeth, Guillemin, Haibao, Haiman, Holm, Huber, Ikeda, Kirillov, Knutson, Kogan, Kostant, Kresh, S. Kumar, A. Kumar, Lam, Lapointe, Lascoux, Lenart, Miller, Morse, Naruse, Peterson, Pitti, Postnikov, Purhboo, Ram, Richmond, Robinson, Shimozono, Sottile, Sturmfels, Tamvakis, Thomas, Vakil, Winkle, Woodward, Yong, Zara...

Some Recommended Further Reading

1. "Schubert Calculus" by Steve Kleiman and Dan Laksov. The American Mathematical Monthly, Vol. 79, No. 10. (Dec., 1972), pp. 1061-1082.

2. "The Symmetric Group" by Bruce Sagan, Wadsworth, Inc., 1991.

3. "Young Tableaux" by William Fulton, London Math. Soc. Stud. Texts, Vol. 35, Cambridge Univ. Press, Cambridge, UK, 1997.

4. "Determining the Lines Through Four Lines" by Michael Hohmeyer and Seth Teller, Journal of Graphics Tools, 4(3):11-22, 1999.

5. "Honeycombs and sums of Hermitian matrices" by Allen Knutson and Terry Tao. Notices of the AMS, February 2001; awarded the Conant prize for exposition.

Some Recommended Further Reading

6. "A geometric Littlewood-Richardson rule" by Ravi Vakil, Annals of Math. 164 (2006), 371-422.

7. "Flag arrangements and triangulations of products of simplices" by Sara Billey and Federico Ardila, Adv. in Math, volume 214 (2007), no. 2, 495–524.

8. "A Littlewood-Richardson rule for two-step flag varieties" by Izzet Coskun. Inventiones Mathematicae, volume 176, no 2 (2009) p. 325–395.

9. "A Littlewood-Richardson Rule For Partial Flag Varieties" by Izzet Coskun. Manuscript. `http://homepages.math.uic.edu/~coskun/`.

10. "Sage:Creating a Viable Free Open Source Alternative to Magma, Maple, Mathematica, and Matlab" by William Stein. `http://wstein.org/books/sagebook/sagebook.pdf`, Jan. 2012.

Generally, these published papers can be found on the web. The books are well worth the money.