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On the derived categories of coherent sheaves on some homogeneous spaces

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To A. Grothendieck on his 60-th birthday

$D^b(P^n)$, the derived category of coherent sheaves on the projective space P^n, was described in the papers of A.A. Beilinson [2] and I.N. Bernstein, I.M. Gelfand, S.I. Gelfand [5]. In the paper [2], two families of sheaves on P^n were distinguished: $\mathcal{O}(-i)$ and $\Omega^i(i)$, $i=0, 1, \ldots, n$. Sheaves from each of these families have no higher Ext's between each other and are "free generators" of the category $D^b(P^n)$. That means, that $D^b(P^n)$ is equivalent (as a triangulated category) to the homotopy category of finite complexes of sheaves, consisting of finite direct sums of $\mathcal{O}(-i)$ (resp. $\Omega^i(i)$), $i=0, 1, \ldots, n$.

In the present paper we describe, in a similar way, some more triangulated categories. More precisely, the description proceeds as follows. If $\mathfrak{A}$ is a pre-additive category, then one can form a triangulated category $\mathrm{Tr}(\mathfrak{A})$, which is "generated freely" by $\mathfrak{A}$. Its objects are finite complexes, consisting of finite formal direct sums of objects of $\mathfrak{A}$, and morphisms are homotopy classes of morphisms of complexes. The description of a given triangulated category $\mathfrak{C}$ in the form $\mathrm{Tr}(\mathfrak{A})$ is practical enough, especially when the functor $\mathfrak{C} \to \mathrm{Tr}(\mathfrak{A})$ is given explicitly. We represent in the form $\mathrm{Tr}(\mathfrak{A})$ the derived categories of coherent sheaves on flag varieties and quadrics, and also the derived categories of finite-dimensional representations of parabolic subgroups in $GL(n, \mathbb{C})$.

In the §1 we fix the notations and recall the formulation of some facts from representation theory and homological algebra, which are necessary for the sequel. Of importance to us is the notion of a convolution, or of a total object of a finite complex over a triangulated category. This notion is needed to make some sense to the words "resolution in the derived category". Such a convolution is not canonical. Moreover, it even not always exists, the obstructions to its existence being the higher Massey compositions (or Toda brackets) of consecutive differentials in the complex. In the context of topological spaces or spectra such questions were treated in [23, 24].

In the §2 we give a general construction of "dual families" (such as $\{\Omega^i(i)\}_{i=0}^n$ for $\{\mathcal{O}(-i)\}_{i=0}^n$), and resolutions of the diagonal. There is also defined a triangulated category $\mathrm{Tr}(\mathfrak{A})$ for a differential graded (DG-) category $\mathfrak{A}$ and, under certain assumptions, a "duality theorem" is proved: $\mathrm{Tr}(\mathfrak{A})^{\mathrm{op}}$ is equivalent to

$\mathrm{Tr}(\hat{\mathfrak{A}})$, where $\hat{\mathfrak{A}}$ is the "cobar-category" for $\mathfrak{A}$. The consideration of DG-models for Ext-algebras (and categories) is a natural generalisation of the Priddy duality [18] between algebras with quadratic relations. This circumstance was indicated to the author by B.L. Feigin and V.V. Schechtman, to whom the author is sincerely grateful. The §2 was also influenced by recent papers of J.-M. Drezet [7] and A.L. Gorodentsev, A.N. Rudakov [8], in which, for each n, a series of pre-additive categories $\mathfrak{A}$ was constructed with the property $\mathrm{Tr}(\mathfrak{A}) \sim D^b(P^n)$.

In §3, the case of flag varieties is treated. In the particular case of Grassmann varieties (as well as for quadrics below, in §4) the corresponding pre-additive category will be a Koszul (in the sense of Priddy [18]) category with quadratic relations. It is not the case for general flag varieties.

In §4, we consider smooth projective quadrics. Our approach is based on the use of the graded Clifford algebra A, which is Priddy dual to the function algebra on the quadratic cone. Besides, we consider flag varieties $F(1, n-1, \mathbb{C}^n)$, which are incidence quadrics. We give for them another representation of the derived category of coherent sheaves in the form $\mathrm{Tr}(\mathfrak{A})$, where the category $\mathfrak{A}$ is Koszul. An approach very close to the ours was developed in the paper of R.G. Swan [19]. Author's result, announced in [12], was obtained independently.

Finally, in §5, the construction of §3 is applied to the category of finite-dimensional representations of parabolic subgroups in $GL(n, \mathbb{C})$. To do this, we consider the homogenous vector bundle on the flag variety, corrresponding to such a representation.

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§1. Preliminaries

1.1. For general facts about triangulated categories, see [10, 21]. The bounded derived category of an abelian category $\mathfrak{A}$ will be denoted $D^b(\mathfrak{A})$. The i times iterated translation functor in a triangulated category will be denoted $E \to E[i]$. On complexes (always supposed to be cohomological) this functor is defined by the formula $(C^\cdot[i])^j = C^{i+j}$. If E, F are objects of a triangulated category $\mathfrak{C}$, then $\mathrm{Ext}^i_{\mathfrak{C}}(E, F)$ means $\mathrm{Hom}_{\mathfrak{C}}(E, F[i])$. If $\mathfrak{A}$ is an additive category, then $\mathrm{Hot}(\mathfrak{A})$ is the homotopy category of bounded complexes over $\mathfrak{A}$. If X is a scheme, then the category of coherent sheaves on X is denoted $\mathrm{Sh}(X)$. The category $D^b(\mathrm{Sh}(X))$ is denoted simply $D^b(X)$. We denote identically algebraic vector bundles and locally free sheaves of their sections.

The category of covariant functors between categories $\mathfrak{C}$ and $\mathfrak{D}$ will be denoted $\mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$, and of contravariant functors-$\mathrm{Fun}^o(\mathfrak{C}, \mathfrak{D})$.

1.2. Let $\mathfrak{C}$ be a triangulated category and $C^\cdot$ – a bounded complex over $\mathfrak{C}$, which we can suppose to be situated in degrees from 0 to n:

$$C^\cdot = \{C^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-1}} C^n\}.$$

A (right) Postnikov system, attached to $C^\bullet$, is, by definition, any diagram

$$
\begin{tikzcd}
C^0[-n] \arrow[dr, "i_0"] & & C^1[-n+1] \arrow[dr, "i_1"] & & \cdots & C^{n-2}[-2] \arrow[dr, "i_{n-2}"] & & C^{n-1}[-1] \arrow[dr, "d^{n-1}"] & & C^n \\
B^0 \arrow[u, "j_{-1}"] & B^1 \arrow[l] \arrow[ur, "j_0"] & & B^2 \arrow[ll] \arrow[ur, "i_1"] & \cdots \arrow[l] & & B^{n-1} \arrow[ll] \arrow[ur, "j_{n-2}"] & & C^n[-1] \arrow[ll] \arrow[ur, "id"] &
\end{tikzcd}
$$

in which all triangles are distinguished and $j_\upsilon \circ i_\upsilon = d^\upsilon$. The morphisms j_υ have the degree $+1$. An object $T \in \mathrm{Ob}\,\mathfrak{C}$ is called a (right) convolution of $C^\bullet$, if there exists a right Postnikov system, attached to $C^\bullet$ such that $T = B^0$. The class of all convolutions of $C^\bullet$ will be denoted $\mathrm{Tot}(C^\bullet)$. Clearly the notions of Postnikov systems and convolutions are stable under exact functors of triangulated categories.

1.3. Remarks and conventions. a) One could also define a left Postnikov system, attached to $C^\bullet$ as a diagram

$$
\begin{tikzcd}
C^0 \arrow[dr, "id"] & & C^1 \arrow[dr, "i_1"] & & C^2 \arrow[dr, "i_2"] & & \cdots & & C^{n-1} \arrow[d] & & C^n \arrow[dr, "i_n"] & \\
& C^0 \arrow[ur, "d^0"] & & D^1 \arrow[ll] \arrow[ur, "j_1"] & & D^2 \arrow[ll] & & \cdots \arrow[ll] \arrow[ur, "i_{n-1}"] & D^{n-1} \arrow[l] \arrow[urr, "j_{n-1}"] & & & D^n \arrow[lll]
\end{tikzcd}
$$

in which $d^\upsilon = j_\upsilon \circ i_\upsilon$, the triangles are distinguished and the horizontal morphisms have the degree $+1$. Then one could define a left convolution of $C^\bullet$ as an object T such that $T = D^n[-n]$ for some left Postnikov system, attached to $C^\bullet$. However, it can be seen using the octohedron axiom, that the class of right convolutions of $C^\bullet$ coincides with the class of left ones. We do not enter into details here, since this will not be used in the present paper. Henceforth all Postnikov systems considered will be right.

b) The class $\mathrm{Tot}(C^\bullet)$ can contain many non-isomorphic elements and can be empty. The condition for its non-emptiness is the vanishing of all Massey products (called also Toda brackets) of every sequence of consecutive differentials in the complex. If one uses the definition of Toda brackets from [23, 24] (which can be trivially adapted to the case of arbitrary triangulated categories), then this fact becomes tautological. In the category $\mathrm{Hot}(\mathfrak{A})$ this definition of Toda brackets coincides with the other possible one, which uses the fact that $\mathrm{Hot}(\mathfrak{A})$ is the cohomology category of the evident differential graded category. A complex over a triangulated category with all the Massey products vanishing will be called true.

c) We shall refer to the terms of a Postnikov system, situated in the lower row, as its B-terms.

1.4. Simple examples of Postnikov systems in the derived category are provided by filtered complexes; then C^i are the filtration quotients and B^i are the terms of the filtration. The Postnikov system, corresponding to the stupid filtration $C^{\geqq i}$ of a complex, will be referred to as its stupid Postnikov system.

Some filtered complexes give rise to Postnikov systems already in the homotopy category. These are so-called twisted complexes [25]. A (bounded) twisted complex over an additive category $\mathfrak{A}$ is a bigraded object $C^{\cdot\cdot}$, endowed with endomorphisms d_i, $i \geqq 0$, of degree $(i, 1-i)$ such that $(\sum d_i)^2 = 0$. We have $d_0^2 = 0$, $d_0 d_1 = -d_1 d_0$, i.e. d_1 defines morphisms of complexes $(C^{i,\cdot}, d_0) \to (C^{i+1,\cdot}, d_0)$. Since $-d_1^2 = d_2 d_0 + d_0 d_2$, the morphism d_1^2 is homotopic to zero. Hence, $\{\dots \to (C^{i,\cdot}, d_0) \to (C^{i+1,\cdot}, d_0) \to \dots\}$ is a complex over $\mathrm{Hot}(\mathfrak{A})$. It is true, and admits a Postnikov system. To see this, we consider the complex $C^{\cdot}$, obtained from $C^{\cdot\cdot}$ by convolution of the grading and equipped with the differential $d = \sum d_i$. It has a filtration "by the second grading degree", which yields a Postnikov system (in $\mathrm{Hot}(\mathfrak{A})$). This system realises $C^{\cdot}$ as a convolution of the said complex over $\mathrm{Hot}(\mathfrak{A})$.

1.5. Let Φ be a cohomological functor from a triangulated category $\mathfrak{E}$ to an abelian category $\mathfrak{A}$. Set $\Phi^p(E) = \Phi(E[p])$ for $E \in \mathrm{Ob}(\mathfrak{E})$ and $p \in \mathbb{Z}$. Let $C^{\cdot}$ be a bounded true complex over E, and $T \in \mathrm{Tot}(C^{\cdot})$. Then there is a spectral sequence $E_1^{pq} = \Phi^q(C^p) \Rightarrow \Phi^{p+q}(T)$. It is constructed starting from a given Postnikov system, realising T as a convolution of $C^{\cdot}$. We delete from this system the term B^0 (in the setting if n.1.1; we assume that $C^{\cdot}$ is situated in degrees from 0 to n for some n). Then we apply to the diagram thus obtained the functor $\Phi^{\cdot}$. Since Φ is a cohomological functor, we get a bigraded exact couple, which yields the desired spectral sequence. The details are left to the reader, since they are entirely analogous to the case of usual spectral sequence of a filtered complex.

1.6. Lemma. *Let $K^{\cdot}$, $L^{\cdot}$ be bounded complexes over an abelian category $\mathfrak{A}$. Suppose that* $\mathrm{Ext}^p_{\mathfrak{A}}(K^i, L^j) = 0$ *for $p > 0$ and all i, j. Then*

$$\mathrm{Hom}_{D^b(\mathfrak{A})}(K^{\cdot}, L^{\cdot}) = \mathrm{Hom}_{\mathrm{Hot}(\mathfrak{A})}(K^{\cdot}, L^{\cdot}).$$

Proof. This can be seen using devissage and the fact that the functor Hom in a triangulated category is cohomological with respect to each argument.

1.7. We shall always, except the beginning of §2, work over the field $\mathbb{C}$ of complex numbers. In fact, one can replace everywhere $\mathbb{C}$ by any field of zero characteristics. The sole reason why we do not do this is that $\mathbb{C}$, being a proper name, has a meaning which is less prone to be forgotten in a long discource.

1.8. Let W be an m-dimensional vector space and $\alpha = (\alpha_1, \dots, \alpha_m)$ be a non-increasing sequence of integers. We shall denote $\Sigma^\alpha W$ the space of the irreducible representation of the group $GL(W)$ with the highest weight α. We shall apply the operation Σ^α as well to vector bundles. There are isomorphisms of $GL(W)$-modules:

$$(\Sigma^{\alpha_1, \dots, \alpha_m} W)^* = \Sigma^{\alpha_1, \dots, \alpha_m}(W^*) = \Sigma^{-\alpha_m, \dots, -\alpha_1}(W)$$

(see [22]). If α is positive, i.e. all $\alpha_i \geqq 0$, then Σ^α can be extend to a covariant polynomial functor from the category of vector spaces to itself [15]. In this case α can be thought of as a Young diagram with α_i as lengths of rows. We shall denote α^* the dual Young diagram and $|\alpha|$ the sum $\Sigma \alpha_i$. If V and W are finite-dimensional vector spaces, then the Cauchy formula ([15], Chap. 1, §4), says that $\Lambda^i(V \otimes W)$ is isomorphic, as $GL(V) \times GL(W)$-module, to the sum $\bigoplus_{|\alpha| = i} \Sigma^\alpha V \otimes \Sigma^{\alpha^*} W$.

1.9. Let α, β be two Young diagrams. The Littlewood-Richardson rule [15] for the decomposition of $\Sigma^\alpha \otimes \Sigma^\beta$ into irreducible factors, prescribes to label each cell of β with the number of the corresponding row. After that, all cells of β must be added to α in such way, as to satisfy the following conditions:

1) for each i, the union of α and all cells with labels less or equal to i, forms a Young diagram;

2) no two cells with equal labels lie in the same column;

3) order the added (i.e. the labelled) cells from the right to the left and from the upper side down. Then, in each initial interval of this sequence, one must have the inequalities: (the number of cells with the label 1)$\geqq$(the number of cells with the label 2)$\geqq \dots$.

The multiplicity of Σ^γ in $\Sigma^\alpha \otimes \Sigma^\beta$ equals the number of admissible ways of adding cells of β to α, which have the type γ and differ by positions of labels. There is also a dual form of this rule, where the cells are labelled according to the number of the corresponding column, rather than row.

It follows from these rules that the number of all rows (resp. columns) in each diagram γ such that $\Sigma^\gamma \subset \Sigma^\alpha \otimes \Sigma^\beta$ can not exceed the sum of such numbers for α and β.

1.10. Let G be a reductive algebraic group, $B \subset G - a$ Borel subgroup, $P \supset B - a$ parabolic one. The vector bundle over G/P, corresponding to a finite-dimensional algebraic representation V of the subgroup P, will be denoted $\tilde{V}$. When V is one-dimensional, i.e. is given by a character $\chi: P \to \mathbb{C}^*$, we shall also use the notation $\mathcal{O}(\chi)$ for $\tilde{V}$. If V is an irreducible representation, then it can be factorized through the Levi subgroup $K \subset P: P \to K \to GL(V)$. The lattices of integral weights of G and K can be identified, so V determines a dominant weight $\chi: T \to \mathbb{C}^*$, where $T \subset B$ is the maximal torus. By the Borel-Weil theorem [13], denoting $p: G/B \to G/P$ the natural projection, one has:

$$R^0 p_* \mathcal{O}(\chi) = \tilde{V}; \quad R^i p_* \mathcal{O}(\chi) = 0, \quad i > 0; \quad H^i(G/P, \tilde{V}) = H^i(G/B, \mathcal{O}(\chi)), \quad i \geqq 0.$$

§ 2. A general construction of dual families and resolutions of diagonal

2.1. Recall that a pre-additive category is a category, in which Hom-sets are endowed with abelian group structure such that the composition is biadditive. A differential graded (DG-) category is a pre-additive category $\mathfrak{A}$, in which abelian groups $\mathrm{Hom}_{\mathfrak{A}}(E, F)$ are equipped with $\mathbb{Z}$-grading and differential d of degree $+1$, $d^2 = 0$, such that for the differential of the composition one has the usual graded Leibnitz formula, and also $\deg(\mathrm{id}_E) = 0$, $d(\mathrm{id}_E) = 0$. We denote $H(\mathfrak{A})$ the cohomology category of $\mathfrak{A}$: $\mathrm{Ob}\, H(\mathfrak{A}) = \mathrm{Ob}\, \mathfrak{A}$; $\mathrm{Hom}_{H(\mathfrak{A})}(E, F) = H^\cdot(\mathrm{Hom}_{\mathfrak{A}}(E, F))$. An important example of a DG-category is provided by the category $C^b(\mathfrak{A})$ of bounded complexes over an additive category $\mathfrak{A}$ (the morphisms are not supposed to commute with differentials). The corresponding cohomology category is $\mathrm{Hot}(\mathfrak{A})$.

2.2. Let $\mathfrak{A}$ be a DG-category, $\mathfrak{A}^\oplus$- the DG-category, obtained from $\mathfrak{A}$ by adjoining formal finite direct sums of objects. Define a new DG-category, Pre-Tr$(\mathfrak{A})$, whose objects are systems $\{(E_i)_{i\in\mathbf{Z}}, q_{ij}: E_i \to E_j\}$, where $E_i \in \mathrm{Ob}(\mathfrak{A}^\oplus)$, almost all E_i are zero; q_{ij} are morphisms in $\mathfrak{A}^\oplus$ of degree $i-j+1$, satisfying the condition $dq_{ij}+\sum_k (q_{kj}q_{ik})=0$ for all i, j. Suppose given two objects of Pre-Tr$(\mathfrak{A})$, $C=\{E_i, q_{ij}\}$ and $C'=\{E'_i, q'_{ij}\}$. The complex $\mathrm{Hom}_{\text{Pre-Tr}(\mathfrak{A})}(C, C')$ has, by definition, the components $\mathrm{Hom}^k = \bigoplus_{l+j-i=k} \mathrm{Hom}^l_{\mathfrak{A}^\oplus}(E_i, E'_j)$. If $f\in \mathrm{Hom}^l_{\mathfrak{A}^\oplus}(E_i, E'_j)$, then its differential in Pre-Tr$(\mathfrak{A})$ is defined by the formula

$$df = d_{\mathfrak{A}^\oplus}(f) + \sum_m (q_{jm} f + (-1)^{i-m+1} f q_{mi}),$$

where $d_{\mathfrak{A}^\oplus}$ is the differential in Hom-groups of $\mathfrak{A}^\oplus$.

Define Tr$(\mathfrak{A})$ to be the category of 0-th cohomology of the DG-category Pre-Tr$(\mathfrak{A})$. It has a natural structure of a triangulated category. In fact, Tr$(\mathfrak{A})$ is a full triangulated subcategory in the homotopy category of contravariant DG-functors $\mathfrak{A} \to \{$complexes of abelian groups$\}$. To define this embedding, denote by h_E for $E\in \mathrm{Ob}\,\mathfrak{A}$ the corresponding representable functor: $F\to \mathrm{Hom}(F, E)$ and $\bar{h}_E$ the functor with values in the category of graded abelian groups, obtained from h_E by forgetting the differential. An object $\{E_i, C_{ij}\}\in \mathrm{Ob}\,\mathrm{Tr}(\mathfrak{A})$ can be identified with the DG-functor, sending $F\in \mathrm{Ob}\,\mathfrak{A}$ to the graded abelian group $\oplus \bar{h}_{E_i}(F)[i]$, equipped with the differential $d+Q$, where $Q=\|q_{ij}\|$, d is the differential in $\oplus h_{E_i}(F)[i]$.

If $\mathfrak{A}'$ is another DG-category and $F: \mathfrak{A}\to\mathfrak{A}'$ is a DG-functor, then one has a DG-functor Pre-Tr(F): Pre-Tr$(\mathfrak{A})\to$ Pre-Tr$(\mathfrak{A}')$ and an exact functor Tr(F): Tr$(\mathfrak{A})\to$ Tr$(\mathfrak{A}')$. If $\mathfrak{A}$ is a pre-additive category with trivial DG-structure, then Tr$(A)=$ Hot$(\mathfrak{A}^\oplus)$. The following proposition goes back at least to [2].

2.3. Proposition. *Let $\mathfrak{A}$ be a full subcategory in an abelian category $\mathfrak{B}$ such that for all $E, F\in \mathrm{Ob}\,\mathfrak{A}$ and $i>0$ one has $\mathrm{Ext}^i_{\mathfrak{B}}(E, F)=0$. Then the inclusion $\mathfrak{A}\subset\mathfrak{B}$ extends to an embedding of* Tr$(\mathfrak{A})=$ Hot$(\mathfrak{A}^\oplus)$ *as a full subcategory into $D^b(\mathfrak{B})$.*

2.4. Let $\mathfrak{C}$ be a triangulated category. We shall denote $\mathfrak{C}^{gr}$ the graded category, made from $\mathfrak{C}$ by the rule: $\mathrm{Ob}\,\mathfrak{C}^{gr}=\mathrm{Ob}\,\mathfrak{C}$, $\mathrm{Hom}_{\mathfrak{C}^{gr}}(E, F)=\oplus \mathrm{Ext}^i_{\mathfrak{C}}(E, F)$. If $\mathfrak{A}\subset\mathfrak{C}^{gr}$ is a full subcategory, then we shall denote $\langle\mathfrak{A}\rangle_{\mathfrak{C}}$ the smallest full triangulated subcategory in $\mathfrak{C}$, containing $\mathfrak{A}$. It is not clear, to what extend $\langle\mathfrak{A}\rangle_{\mathfrak{C}}$ is defined by $\mathfrak{A}$ itself with its structure of a "category with Massey products". In particular, it is not clear, whether one can replace in Proposition 2.3. $D^b(\mathfrak{B})$ by an arbitrary triangulated category.

2.5. Henceforth all DG-categories are supposed to be $\mathbf{C}$-linear. Vect denotes the category of finite-dimensional vector spaces over $\mathbf{C}$.

Call a DG-category $\mathfrak{A}$ ordered, if $\mathrm{Ob}\,\mathfrak{A}$ is a set, endowed with a partial order $\leq$ such that $\mathrm{Hom}(E, F)=0$ unless $E\leq F$ and $\mathrm{Hom}(E, E)=\mathbf{C}$ (in the degree 0) for all $E, F\in \mathrm{Ob}\,\mathfrak{A}$. Call a DG-category $\mathfrak{A}$ finite, if $\mathrm{Ob}\,\mathfrak{A}$ is a finite set, and all $\mathrm{Hom}_{\mathfrak{A}}(E, F)$ are bounded complexes of finite-dimensional vector spaces.

2.6. Proposition. *Let $\mathfrak{A}$, $\mathfrak{A}'$ be finite ordered DG-categories, $f: \mathfrak{A}\to\mathfrak{A}'$ – a DG-functor such that $H(f)$ is an isomorphism of categories. Then* Tr(f): Tr$(\mathfrak{A})\to$ Tr$(\mathfrak{A}')$ *is an equivalence.*

Proof. First show that for all $C_1, C_2 \in \mathrm{Ob}\,\mathrm{Tr}(\mathfrak{A})$ the morphism

$$(\mathrm{Tr}\,f)_*: \mathrm{Hom}_{\mathrm{Tr}(\mathfrak{A})}(C_1, C_2) \to \mathrm{Hom}_{\mathrm{Tr}(\mathfrak{A}')}((\mathrm{Tr}\,f)(C_1), (\mathrm{Tr}\,f)(C_2))$$

is an isomorphism. By our assumption, it is true when C_1 and C_2 have the form $E[i]$, $E \in \mathrm{Ob}\,\mathfrak{A}$. It follows from the orderedness and finiteness of $\mathfrak{A}$, that every object of $\mathrm{Tr}(\mathfrak{A})$ can be constructed, starting from the simplest objects $E[i]$, $E \in \mathrm{Ob}\,\mathfrak{A}$, by applying successively the operation of taking mapping cone. Hence, the devissage yields the isomorphicity of $(\mathrm{Tr}\,f)_*$ and $\mathrm{Im}\,\mathrm{Tr}\,f$ is a full triangulated subcategory in $\mathrm{Tr}(\mathfrak{A}')$. Again using the devissage, we find that this subcategory is in fact equivalent to whole $\mathrm{Tr}(\mathfrak{A}')$.

2.7. Let $\mathfrak{A}$ be a finite ordered DG-category. Define a new DG-category, $\hat{\mathfrak{A}}$, whose objects are symbols $\hat{E}$ for $E \in \mathrm{Ob}\,\mathfrak{A}$. Denote the complexes $\mathrm{Hom}_{\mathfrak{A}}(E, F)$ simply $[E, F]$. Set $\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F}) = 0$ unless $E \leqq F$, $\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{E}) = \mathbb{C}$, and for $E < F$ (i.e. $E \leqq F$ and $E \neq F$) define $\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})$ to be equal to the total complex of the double complex

$$[E, F]^* \to \bigoplus_{E<G<F} [E, G]^* \otimes [G, F]^* \to$$
$$\to \bigoplus_{E<G_1<G_2<F} [E, G_1]^* \otimes [G_1, G_2]^* \otimes [G_2, F]^* \to \dots \tag{2.1}$$

which is a kind of bar-construction [16]. The horizontal grading of this double complex assigns to $[E, F]^*$ the degree 1, to the next sum – the degree 2, etc. We omit the obvious formulae for differentials in (2.1), as well as in the similar complexes below.

$\hat{\mathfrak{A}}$ is also a finite ordered DG-category; there is a natural DG-functor $i: \hat{\mathfrak{A}} \to \mathfrak{A}$, sending each $\hat{E}$ to E, which is a quasi-isomorphism (on Hom-complexes).

2.8. *Remarks.* 1) Since $\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})$ is the total complex of a natural double complex, the spaces $H^i\,\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})$ have an additional grading.

2) If $\mathfrak{A}$ is an ordered category with trivial DG-structure, and the poset $\mathrm{Ob}\,\mathfrak{A}$ is ranked with rank function r, then one can introduce a grading into $\mathrm{Mor}(\mathfrak{A})$, ascribing to $\mathrm{Hom}(E, F)$ the degree $r(E) - r(F)$.

3) Suppose, in the setting of 2.7, that the differentials in $\mathfrak{A}$ are trivial. Then, with respect to the bigrading of Remark 1.

$$H^{i,\cdot}\,\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F}) = \mathrm{Ext}^i_{\mathrm{Fun}(\mathfrak{A}, \mathrm{Vect})}(\mathbb{C}_E, \mathbb{C}_F),$$

where $\mathbb{C}_E$ is the functor, sending E to $\mathbb{C}$, all other objects-to 0 (see also n.2 below). If one has only $H^{i,i}$ non-vanishing, i.e. Ext^i has, with respect to the additional grading degree i, then we shall, following [18, 24], call the category $\mathfrak{A}$ Koszul. The proof of Theorem 1.2 from [14] applies as well to this case and yields that $\mathrm{Mor}(\mathfrak{A})$ is generated by morphisms of degree 1, and relations between such morphisms follow from quadratic relations. Examples of such categories are provided by full subcategories in $\mathrm{Sh}(P^n)$, whose sets of objects are "triads" of Drezet [7] or "bases of helices" of Gorodentsev and Rudakow [8]. Other examples see in § 3, 4 below.

4) Let $R(\mathfrak{A})$ be the square matrix, indexed by the set $\mathrm{Ob}\,\mathfrak{A}$ with matrix elements $R(\mathfrak{A})_{E,F}=\chi(\mathrm{Hom}_{\mathfrak{A}}(E, F))$ (Euler characteristic). It is an upper-triangular (with respect to the given ordering) matrix with units on the diagonal. One has the relation $R(\hat{\mathfrak{A}})=R(\mathfrak{A})^{-1}$, since on the level of Euler characteristics the complex (2.1) gives the geometric progression: $(1+N)^{-1}=1-N+N^2-\dots$. Here $N=R(\mathfrak{A})-1$.

2.9. Let $\mathfrak{A}$ be a finite ordered DG-category, $\mathfrak{A}^{\mathrm{op}}$-its opposite category; we shall think of objects of $\mathfrak{A}^{\mathrm{op}}$ as symbols E^* for $E\in\mathrm{Ob}\,\mathfrak{A}$. For $E=\mathrm{Ob}\,\mathfrak{A}$ consider the object $\phi(\hat{E})$ of the category $\mathrm{Pre\text{-}Tr}(\mathfrak{A}^{\mathrm{op}})$, defined by the complex

$$\underset{-2}{\dots\to\bigoplus_{E<E_1<E_2}[E, E_1]\otimes[E_1, E_2]\otimes E_2^*}\to\underset{-1}{\bigoplus_{E<E_1}[E, E_1]\otimes E_1^*}\to\underset{0}{E^*}$$

which is also a kind of bar-construction. The notation of the type $[E, E_1]\otimes E^*$ means that we have chosen bases in all $[E, F]$ and consider the corresponding direct sums of objects.

2.10. Proposition. *The correspondence $\hat{E}\to\phi(\hat{E})$ extends to a DG-functor ϕ:* $\mathrm{Pre\text{-}Tr}(\hat{\mathfrak{A}})\to\mathrm{Pre\text{-}Tr}(\mathfrak{A}^{\mathrm{op}})$. *Its cohomology functor $H(\phi)$:* $\mathrm{Tr}(\hat{\mathfrak{A}})\to\mathrm{Tr}(\mathfrak{A}^{\mathrm{op}})$ *is an equivalence of triangulated categories.*

Proof. First, we construct a functor $\phi_0\colon\hat{\mathfrak{A}}\to\mathrm{Pre\text{-}Tr}(\mathfrak{A}^{\mathrm{op}})$, sending $\hat{E}$ to $\phi(\hat{E})$. The desired morphisms of complexes $\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})\otimes\phi(\hat{E})\to\phi(\hat{F})$ will be defined from morphisms $\phi(\hat{E})\to\phi(\hat{F})\otimes\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})^*$ by partial dualisation. To define these, note that the complex $\phi(\hat{F})\otimes\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})^*$, if regarded as vector space, consists, like $\phi(\hat{E})$, of summands corresponding to chains $E<E_1<\dots$, which start from E, but, unlike $\phi(\hat{E})$, necessarily pass through F. Also, in the formula for the differential in $\phi(\hat{F})\otimes\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})^*$ the summand corresponding to the subchain with F omitted, is not included into summation. Define a morphism $\phi(\hat{E})\to\phi(\hat{F})\otimes\mathrm{Hom}_{\hat{\mathfrak{A}}}(\hat{E}, \hat{F})^*$ componentwise, sending the summand corresponding to a chain $E<E_1<\dots$ to itself, if the chain passes through F, and to zero, otherwise. The functor ϕ_0 is constructed.

If $C=\{\hat{E}_i, q_{ij}\}$ is an object of $\mathrm{Pre\text{-}Tr}(\mathfrak{A})$, then applying the functor ϕ_0 to each component, we obtain a complex of objects of $\mathrm{Pre\text{-}Tr}(\mathfrak{A})$. Taking its natural convolution, we obtain an object $\phi(C)$. This defines the desired DG-functor ϕ. Consider its square

$$\phi^2\colon\mathrm{Pre\text{-}Tr}(\hat{\hat{\mathfrak{A}}}{}^{\mathrm{op}})\to\mathrm{Pre\text{-}Tr}(\hat{\mathfrak{A}})\to\mathrm{Pre\text{-}Tr}(\mathfrak{A}^{\mathrm{op}}).$$

There is a natural transformation $\phi^2\to\mathrm{Pre\text{-}Tr}(i)$, where $i\colon\hat{\hat{\mathfrak{A}}}{}^{\mathrm{op}}\to\mathfrak{A}^{\mathrm{op}}$ is the natural quasi-isomorphism. This transformation sends each object $\phi(\hat{E}^*)$ to E^* and gives the identity on the cohomology of Hom-complexes. The proposition follows from this.

Henceforth, we shall denote $\phi(\hat{E})$ simply $\hat{E}$, identifying $\mathrm{Tr}(\hat{\mathfrak{A}})$ and $\mathrm{Tr}(\mathfrak{A}^{\mathrm{op}})$.

2.11. There is another triangulated category connected with a DG-category $\mathfrak{A}$, namely, the derived category $D^b(\mathfrak{A})$. Let us recall the construction, supposing for simplicity that $\mathfrak{A}$ is finite. First consider the homotopy category of contravariant DG-functors $\mathfrak{A} \to \mathrm{Vect}$, which will be denoted $\mathrm{Hot}^b(\mathfrak{A})$. Then localise it with respect to the family of quasi-isomorphisms, i.e. those natural transformations of functors $\phi \to \psi$, which give a quasi-isomorphism of complexes $\phi(E) \to \psi(E)$ for every object $E \in \mathrm{Ob}(\mathfrak{A})$. For example, if the DG-structure in $\mathfrak{A}$ is trivial, then $D^b(\mathfrak{A})$ is the bounded derived category of the abelian category $\mathrm{Fun}^o(\mathfrak{A}, \mathrm{Vect})$.

2.12. Proposition. *Let* $P \in \mathrm{Ob}\, D^b(\mathfrak{A})$ *have the form* $\bigoplus_{E \in \mathrm{Ob}\,\mathfrak{A}} h_E \otimes C^{\bullet}_E$, *where* h_E *are the representable functors and* $C^{\bullet}_E$ *are bounded complexes over Vect. Then, for any object* $\psi \in \mathrm{Ob}\, D^b(\mathfrak{A})$ *the natural morphism* $\mathrm{Hom}_{\mathrm{Hot}^b(\mathfrak{A})}(P, \psi) \to \mathrm{Hom}_{D^b(\mathfrak{A})}(P, \psi)$ *is an isomorphism.*

Proof. It is similar to the case of trivial DG-structure in $\mathfrak{A}$ (then the representable functors are projective objects in $\mathrm{Fun}^o(\mathfrak{A}, \mathrm{Vect})$). It suffices to consider only the case $P = h_E$. As in the "classical" case, the proposition follows from the following lemma, whose proof is left to the reader.

Lemma. *Any morphism from* h_E *to an (objectwise) acyclic DG-functor is homotopic to zero.*

We have natural exact functors

$$\mathrm{Tr}(\mathfrak{A}) \xrightarrow{\alpha} \mathrm{Hot}^b(\mathfrak{A}) \xrightarrow[\beta]{} D^b(\mathfrak{A}),$$

where α is the functor, spoken of in n.2.1.

2.13. Proposition. *If* $\mathfrak{A}$ *is a finite ordered DG-category, then the composite functor* $\beta\alpha$ *is an equivalence between* $\mathrm{Tr}(\mathfrak{A})$ *and* $D^b(\mathfrak{A})$.

Proof. It suffices to show that any contravariant DG-functor $\psi\colon \mathfrak{A} \to C^b(\mathrm{Vect})$ is quasi-isomorphic to a DG-functor, admitting a Postnikov system over $\mathrm{Hot}^b(\mathfrak{A})$ with "factors" of the form $\oplus h_E \otimes C^{\bullet}_E$. To do this, consider the bar-resolution of the functor ψ. It is a DG-functor, sending an object $E \in \mathrm{Ob}\,\mathfrak{A}$ to the total complex of the following double complex

$$\dots \to \bigoplus_{E < F_1 < F_2} [E, F_1] \otimes [F_1, F_2] \otimes \psi(F_2) \to \bigoplus_{E < F} [E, F] \otimes \psi(F).$$

It is easily seen to be quasi-isomorphic to ψ. On the other hand, each functor $E \to [E, F_1] \otimes [F_1, F_2] \otimes \dots \otimes \psi(F_n)$ satisfies the conditions of Proposition 2.12. Taking the filtration of the bar-complex by the horizontal degree, we obtain the required Postnikov system.

2.14. Now suppose given a smooth complete algebraic variety M. Suppose that a finite ordered pre-additive category $\mathfrak{A}$ (with trivial DG-structure) is given as a full subcategory in $D^b(M)^{\mathrm{gr}}$ and objects of $\mathfrak{A}$ are locally free sheaves. In other words, $\mathrm{Hom}_{\mathfrak{A}}(E, F) = \mathrm{Hom}_M(E, F)$, $\mathrm{Ext}^i_M(E, F) = 0$ for $i > 0$ and all $E, F \in \mathrm{Ob}\,\mathfrak{A}$. Then we can consider complexes $\hat{E}$ for $E \in \mathrm{Ob}\,\mathfrak{A}$ as objects of $D^b(M)$, understanding * as dualisation. In this situation we have the following orthogonality relation for hypercohomology.

2.15. Proposition. *For all $E, F\in \mathrm{Ob}\,\mathfrak{A}$*

$$\mathbb{H}^p(M, E\otimes \hat{F})=\begin{cases}0 & \textit{for all } p, \quad \textit{if } E\neq F\\ 0 & \textit{for } p\neq 0, \quad \textit{if } E=F\\ \mathbb{C} & \textit{for } p=0, \quad \textit{if } E=F\end{cases}$$

Proof. $E\otimes \hat{F}$ is the complex

$$\ldots\to \bigoplus_{F<F_1} E\otimes F_1^*\otimes[F, F_1]\to E\otimes F^*.$$

Hence, there is a spectral sequence, converging ho $\mathbb{H}^\cdot(M, E\otimes \hat{F})$, whose first term is the sum of summands $[F, F_1]\otimes\ldots\otimes[F_{k-1}, F_k]\otimes[F_k, E]$, corresponding to chains $F<F_1<\ldots, F_k\leq E$. Define in this first term a homotopy, sending each such term to zero, if $F_k=E$, and by identity to the summand corresponding to $F<F_1<\ldots<F_k<F_{k+1}=E$, if $F_k\neq E$. This homotopy shows that our spectral sequence degenerates in the first term and we obtain the claimed result.

Example. If $M=P^n$, $\mathrm{Ob}\,\mathfrak{A}=\{\mathcal{O}(-i), i=0, 1, \ldots, n\}$, then $\mathcal{O}(i)^\wedge$ is quasi-isomorphic to $\Omega^i(i)[-i]$.

2.16. Consider, in the setup of n.2.14, the following complex $B^\cdot=B^\cdot_{\mathfrak{A}}$ of sheaves on $M\times M$:

$$\ldots\to \bigoplus_{E<E_1<E_2} E\boxtimes E_2^*\otimes[E, E_1]\otimes[E_1, E_2]\to$$
$$\to \bigoplus_{E<E_1} E\boxtimes E_1^*\otimes[E, E_1]\to \bigoplus_{E\in \mathrm{Ob}\,\mathfrak{A}} E\boxtimes E^*$$

where $\boxtimes$ is the "external" tensor product of sheaves. There is a natural morphism $B^\cdot\to\mathcal{O}_\Delta$, where $\Delta\subset M\times M$ is the diagonal. It is, of cource, not always a quasi-isomorphism. Nevertheless, this construction covers the resolutions of diagonal, considered in [7, 8, 11, 12].

For each complex $\mathcal{G}\in\mathrm{Ob}\,D^b(M\times M)$ denote $\Psi_{\mathcal{G}}$ the functor

$$Rp_{2*}(p_1^*(-)\overset{\mathbf{L}}{\otimes}\mathcal{G}): D^b(M)\to D^b(M).$$

Here p_i: $M\times M\to M$ are the projections.

2.17. Proposition. *For all $\mathcal{F}^\cdot\in\mathrm{Ob}\,\mathrm{Tr}(\mathfrak{A})\subset\mathrm{Ob}\,D^b(M)$ the morphism $\Psi_B(\mathcal{F}^\cdot)\to\mathcal{F}^\cdot$, induced by the morphism $B^\cdot\to\mathcal{O}_\Delta$, is a quasi-isomorphism.*

Proof. It suffices to consider the case $\mathcal{F}^\cdot=E\in\mathrm{Ob}\,\mathfrak{A}$. We have a spectral sequence, converging to the cohomology sheaves of the complex $\Psi_B(E)$, whose first term consists of summands of the form $E_1\otimes[E_1, E_2]\otimes\ldots\otimes[E_k, E]$ for $E<E_1<\cdots<E_k\leq E$. The homotopy, similar to that of 2.15, yields the result. □

2.18. Suppose, in addition, that the poset $\mathrm{Ob}\,\mathfrak{A}$ is ranked with rank function r. Then we can consider the following complex over $D^b(M\times M)$:

$$\ldots\to \bigoplus_{r(E)=r_0-2} E\boxtimes\hat{E}[2]\to \bigoplus_{r(E)=r_0-1} E\boxtimes\hat{E}[1]\to \bigoplus_{r(E)=r_0} E\boxtimes\hat{E}$$

where r_0 is the maximal value of the function r. This is a true complex. It admits a Postnikov system due to the fact that $B^{\cdot}_{\mathfrak{A}}$ is in this case a twisted complex. This Postnikov system realises $B_{\mathfrak{A}}$ as a convolution of the said complex.

§3. Grassmanians and flag varieties

3.1. Recall the description of the derived category of coherent sheaves on Grassmanians, since this particular case will be needed for consideration of general flag varieties. Let V be a n-dimensional vector space, $G=G(k, V)$ – the Grassmanian of k-dimensional subspaces in V. $S\subset V$ – the tautological k-dimensional vector bundle over G, $S^{\dagger}=(\tilde{V}/S)^{*}\subset\tilde{V}^{*}$ – its orthogonal complement. Then $H^0(G, S^*)=V^*$; $H^0(G, \tilde{V}/S)=V$. The section $s\in H^0(G\times G, S^*\boxtimes\tilde{V}/S)=V^*\otimes V=\operatorname{End}(V)$, corresponding to the unit operator, vanishes exactly along the diagonal $\Delta\subset G\times G$ and defines the Koszul resolution

$$\{\ldots\to\Lambda^2(S\boxtimes S^{\dagger})\to S\boxtimes S^{\dagger}\to\mathcal{O}_{G\times G}\}=C^{\cdot}$$

of the sheaf $\mathcal{O}_{\Delta}$. Its i-th term is the sum $\oplus\Sigma^{\alpha}S\boxtimes\Sigma^{\alpha^*}S^{\dagger}$, where α runs through Young diagrams with i cells. This resolution yields the generalised Beilinson spectral sequence

$$E_1^{pq}=\bigoplus_{|\alpha|=-p}\mathbb{H}^q(G, \mathcal{F}^{\cdot}\otimes\Sigma^{\alpha^*}S^{\dagger})\otimes\Sigma^{\alpha}S\Rightarrow\underline{H}^{p+q}(\mathcal{F}^{\cdot})$$

where $\mathcal{F}^{\cdot}\in\operatorname{Ob}D^b(G)$ is an arbitrary complex. The following lemma is proved (perhaps, not for the first time) in [11].

3.2. Lemma. a) *Suppose $\gamma_1\geqq\gamma_2\geqq\ldots\geqq\gamma_k\geqq-(n-k)$. Then the sheaf $\Sigma^{\gamma_1,\ldots,\gamma_k}(S^*)$ on $G(k, V)$ has no higher cohomology groups. It can have non-trivial H^0 only when all $\gamma_i\geqq 0$. In this case $H^0(G, \Sigma^{\gamma_1,\ldots,\gamma_k}(S^*))=\Sigma^{\gamma_1,\ldots,\gamma_k}(V^*)$.*

b) *If $(n-k)\geqq\alpha_1\geqq\ldots\geqq\alpha_k\geqq 0$ and $k\geqq\beta_1\geqq\ldots\geqq\beta_{n-k}\geqq 0$, then, denoting $\alpha=(\alpha_1,\ldots,\alpha_k)$, $\beta=(\beta_1,\ldots,\beta_{n-k})$, one has*

$$H^i(G, \Sigma^{\alpha}S\otimes\Sigma^{\beta}S^{\dagger})=\begin{cases}0 & \text{for all } i, \quad \text{if } \alpha\neq\beta^*\\ 0 & \text{for } i\neq|\alpha|=|\beta|, \quad \text{if } \alpha=\beta^*\\ \mathbb{C} & \text{for } i=|\alpha|=|\beta|, \quad \text{if } \alpha=\beta^*.\end{cases}\qquad\square$$

3.3. Denote $X(k, n)$ the set of sheaves $\Sigma^{\alpha}S$ on G, where a runs over Young diagrams with no more than k rows and no more than $n-k$ columns. It follows from the Littlewood-Richardson rule that if $\Sigma^{\alpha}S$, $\Sigma^{\beta}S\in X(k, n)$, then each irreducible summand $\Sigma^{\gamma}S\subset\underline{\operatorname{Hom}}(\Sigma^{\alpha}S, \Sigma^{\beta}S)$ satisfies the conditions of n.a) of the lemma. Hence sheaves from $X(k, n)$ have no higher Ext's between each other. It allows us to construct, following [3], not merely a spectral sequence, but a genuine (two-sided) resolution of arbitrary $\mathcal{F}^{\cdot}\in\operatorname{Ob}D^b(G)$ by means of $\Sigma^{\alpha}S\in X(k, n)$, which is canonical up to homotopy equivalence. Namely, denote $p_i\colon G\times G\to G$ the projections, $i=1, 2$. Then $\mathcal{F}^{\cdot}$ is a convolution of the following true complex over $D^b(G)$

$$\{\ldots \to \bigoplus_{|\alpha|=i} R\Gamma(G, \mathscr{F}^{\cdot} \otimes \Sigma^{\alpha^*} S^{\dagger}) \otimes \Sigma^{\alpha} S \to \ldots\}. \tag{3.1}$$

This complex comes equipped with a canonical Postnikov system, realising $\mathscr{F}^{\cdot}$ as its convolution. Namely, we start from the stupid Postnikov system of the Koszul complex $C^{\cdot}$, which realises $\mathcal{O}_{\Delta}$ as its convolution (in the derived category). Then we tensor this system termwise with the object $p_1^* \mathscr{F}^{\cdot}$ and after that apply to the obtained Postnikov system the exact functor Rp_{2*}. We take in (3.1) for $R\Gamma$ the graded spaces of hypercohomology endowed with zero differentials. The i-th term of (3.1) thus understood will be denoted Z^i.

Now we shall pick nice representatives in Hot(Sh(G)) also for the B-terms (see 1.3) of our Postnikov system, i.e., for the partial convolutions of (3.1), starting from the left. To begin, we consider the differential $Z^{-1} \to Z^0$. It is already represented by a genuine morphism of complexes, and we define the complex B^{-1} to be the cone of this morphism. Since we have done nothing but chosen a nice representative for a term of already existing Postnikov system in the derived category, we dispose with a morphism $Z^{-2} \to B^{-1}$ in $D^b(G)$. The terms of these complexes are direct sums of $\Sigma^{\alpha} S \in X(k, n)$ and therefore have no higher Ext's between each other. Hence, by Lemma 1.6, the morphism $Z^{-2} \to B^{-1}$ can be represented by a genuine homotopy class of morphisms, and we define a complex B^{-2} to be the cone of any morphism from this class. Again, terms of B^{-2} are direct sums of elements of $X(k, n)$ and we dispose with a morphism $Z^{-3} \to B^{-2}$ in the derived category. Thus proceeding, we obtain a complex of sheaves, quasi-isomorphic to $\mathscr{F}^{\cdot}$, whose p-th term equals

$$\bigoplus_{i-j=p} \bigoplus_{|\alpha|=j} \mathbb{H}^i(G, \mathscr{F}^{\cdot} \otimes \Sigma^{\alpha^*} S^{\dagger}) \otimes \Sigma^{\alpha} S.$$

While its terms are defined by $\mathscr{F}^{\cdot}$ in a canonical way, its differentials are not canonically defined by $\mathscr{F}^{\cdot}$; only the homotopy equivalence class of the complex is canonical.

These considerations lead to the following description of $D^b(G)$. Denote $\mathfrak{A}(k, V)$ the full subcategory of $Sh(G(k, V))$ on the objects from $X(k, n)$.

3.4. Theorem. *$D^b(G(k, V))$ is equivalent to* Tr$(\mathfrak{A}(k, V))$ *as a triangulated category.*

3.5. $\mathfrak{A}(k, V)$ is a finite ordered pre-additive category: Hom$(\Sigma^{\alpha} S, \Sigma^{\alpha} S) = \mathbb{C}$ and Hom$(\Sigma^{\alpha} S, \Sigma^{\beta} S) \neq 0$ only if $\alpha_i \geqq \beta_i$ for all i, what will be denoted $\alpha \geqq \beta$. Hence as noted in n.2.13, Tr$(\mathfrak{A}(k, V))$ coincides with $D^b(\mathrm{Fun}^0(\mathfrak{A}(k, V), \mathrm{Vect})$, and we obtain a non-standard t-structure (in the sense of [4]) on the triangulated category $D^b(G(k, V))$ with the heart $\mathrm{Fun}^0(\mathfrak{A}(k, V), \mathrm{Vect})$.

Identifying $G(k, V)$ with $G(n-k, V^*)$, we find that $D^b(G(k, V))$ is equivalent to Tr$(\mathfrak{A}(n-k, V^*)$. The poset $X(k, n)$ is ranked with rank function $|\alpha|$. If $E = \Sigma^{\alpha} S \in \mathrm{Ob}\, \mathfrak{A}(k, V)$, then the complex $\hat{E}$, defined as in §2, is quasi-isomorphic, due to 3.2.b), 2.15 and the generalised Beilinson spectral sequence, to the sheaf $\Sigma^{\alpha^*} S^{\dagger}$, shifted by $|\alpha|$ places to the left. The resolution $C^{\cdot}$ is therefore a particular case of the complex from 2.18.

Denote $\mathfrak{A}_g(k, V)$ the graded category, obtained from $\mathfrak{A}(k, V)$ by assigning to Hom $(\Sigma^\alpha S, \Sigma^\beta S)$ the degree $|\alpha|-|\beta|$, Then the DG-category $\mathfrak{A}_g(k, V)^\wedge$, defined according to n.2.7, is quasi-isomorphic to $\mathfrak{A}(n-k, V^*)$. Since all higher Ext's between objects of $\mathfrak{A}(n-k, V^*)$ also vanish, we obtain by [14] the following fact.

3.6. Corollary. *$\mathfrak{A}_g(k, V)$ and $\mathfrak{A}_g(n-k, V^*)$ are Koszul categories with quadratic relations, Priddy dual to each other.* $\square$

3.7. Let V be a rank n vector bundle over an algebraic variety Y, $G=G(k, V)$ - the relative Grassmanian, $\pi\colon G\to Y$, $p_i\colon G\underset{Y}{\times} G\to G$, $i=1, 2$ - the projections, $\Delta\subset G\underset{Y}{\times} G$ - the relative diagonal. The previous considerations are easily globalised, giving for each $\mathscr{F}^\bullet\in\operatorname{Ob} D^b(G)$ a true complex over $D^b(G)$:

$$\ldots\to\bigoplus_{|\alpha|=i}\pi^*(R\pi_*(\mathscr{F}^\bullet\otimes\Sigma^{\alpha^*}S^\dagger))\otimes\Sigma^\alpha S\to\ldots$$

equipped with a canonical Postnikov system, realising $\mathscr{F}^\bullet$ as its convolution.

3.8. Now we turn to the case of general flag varieties. Suppose given $1\leqq i_1<\ldots<i_k\leqq n$ and let $F=F(i_1, \ldots, i_k, n)$ be the variety of flags of type $(i_1, \ldots, i_k)$ in an n-dimensional vector space V, $S_{i_1}\subset\ldots\subset S_{i_k}\subset V$ - the tautological flag of bundles over F. Denote $X=X(i_1, \ldots, i_k, n)$ the set of sheaves $\Sigma^{\alpha_1}S_{i_1}\otimes\ldots\otimes\Sigma^{\alpha_k}S_{i_k}$ over F, where α_j, $j=1, \ldots, k-1$, runs over Young diagrams with no more than i_j rows, and no more than $i_{j+1}-i_j$ columns; α_k runs over Young diagrams with $\leqq i_k$ rows and $\leqq n-i_k$ columns.

3.9. Proposition. *Sheaves from the set $X(i_1, \ldots, i_k, n)$ have no higher Ext's between each other.*

Proof. We have to calculate the cohomology of sheaves

$$\underline{\operatorname{Hom}}(\Sigma^{\alpha_1}S_{i_1}, \Sigma^{\beta_1}S_{i_1})\otimes\ldots\otimes\underline{\operatorname{Hom}}(\Sigma^{\alpha_k}S_{i_k}, \Sigma^{\beta_k}S_{i_k})$$

on F, where Young diagrams α_j, β_j satisfy the above conditions. From the Littlewood-Richardson rule we have that every irreducible summand $\Sigma^{\gamma_1}S_{i_1}$ in $\underline{\operatorname{Hom}}(\Sigma^{\alpha_1}S_{i_1}, \Sigma^{\beta_1}S_{i_1})$ has components $\gamma_{1,j}$ satisfying the inequalities $-i_2+i_1\leqq\gamma_{1,j}\leqq i_2-i_1$, every summand $\Sigma^{\gamma_2}S_{i_2}$ in $\underline{\operatorname{Hom}}(\Sigma^{\alpha_2}S_{i_2}, \Sigma^{\beta_2}S_{i_2})$ satisfies the inequality $-i_3+i_2\leqq\gamma_{2,j}\leqq i_3-i_2$, etc. It suffices to prove that every sheaf $\bigotimes_{j=1}^{k}\Sigma^{\gamma_j}S_{i_j}$ on $F(i_1, \ldots, i_k, V)$ such that the components $\gamma_{j,p}$ (possibly negative) satisfy the inequalities $\gamma_{1,p}\leqq i_2-i_1$, $\gamma_{2,p}\leqq i_3-i_2$, $\ldots$, $\gamma_{k,p}\leqq n-i_k$, has no higher cohomology. We shall prove this by induction on k, considering the (higher) direct images of this sheaf in the tower of projections

$$F(i_1, i_2, \ldots, i_k, V)\to F(i_2, i_3, \ldots, i_k, V)\to\ldots\to G(i_k, V).$$

We claim that at each step only the 0-th direct image R^0 can be non-trivial. For the first projection (to $F(i_2, \ldots, i_k, V)$) this follows from Lemma 3.2a) and

R^0 equals to
$$\Sigma^{-\gamma_{1,i_1},\dots,-\gamma_{1,1}}(S_{i_2}^*)\otimes\Sigma^{\gamma_2}S_{i_2}\otimes\dots\otimes\Sigma^{\gamma_k}S_{i_k}$$
(if $-\gamma_{1,i_1}\geqq\dots\geqq-\gamma_{1,1}\geqq 0$; otherwise, $R^0=0$). Decomposing this into irreducible summands, we obtain a sum of bundles of the form $\Sigma^{\gamma'_2}S_{i_2}\otimes\Sigma^{\gamma_3}S_{i_3}\otimes\dots\otimes\Sigma^{\gamma_k}S_{i_k}$, where γ'_2 again satisfies the inequalities $\gamma'_{2,p}\leqq(i_3-i_2)$. Thus proceeding, we obtain the desired result. □

Denote $\mathfrak{A}=\mathfrak{A}(i_1,\dots,i_k,n)$ the full subcategory in $\mathrm{Sh}(F(i_1,\dots,i_k,V))$ on the set of objects $X(i_1,\dots,i_k,n)$.

3.10. Theorem. *$D^b(F(i_1,\dots,i_k,V))$ is equivalent to $\mathrm{Tr}(\mathfrak{A}(i_1,\dots,i_k,V))$ as a triangulated category.*

Proof. From the Proposition 2.3, we have an embedding $\mathrm{Tr}(\mathfrak{A})\subset D^b(F)$ as a full triangulated subcategory. It is enough to prove, therefore, that each $\mathscr{F}^\cdot\in\mathrm{Ob}\,D^b(F)$ admits a resolution belonging to $\mathrm{Tr}(\mathfrak{A})$. We shall consider for simplicity the case $k=2$. The general case can be considered similarily, by induction. Denote $p\colon F(i_1,i_2,V)\to G=G(i_2,V)$ the projection. For each α_1 the complex $Rp_*(\mathscr{F}^\cdot\otimes\Sigma^{\alpha_1^*}(S_{i_2}/S_{i_1})^*)$ of sheaves on G has a resolution consisting of direct sums of $\Sigma^{\alpha_2}S_{i_2}$ here α_1 has $\leqq(i_2-i_1)$ columns, α_2 has $\leqq(n-i_2)$ columns). The terms of this resolution are sums of summands of the form
$$\begin{aligned}&\mathbb{H}^i(G,Rp_*(\mathscr{F}^\cdot\otimes\Sigma^{\alpha_1^*}(S_{i_2}/S_{i_1})^*)\otimes\Sigma^{\alpha_2^*}(S_{i_2}^\dagger))\otimes\Sigma^{\alpha_2}S_{i_2}\\&=\mathbb{H}^i(F,\mathscr{F}^\cdot\otimes\Sigma^{\alpha_1^*}(S_{i_2}/S_{i_1})^*\otimes\Sigma^{\alpha_2^*}(S_{i_2}^\dagger))\otimes\Sigma^{\alpha_2}S_{i_2}.\end{aligned}$$
We can identify, in the derived category, $Rp_*(\mathscr{F}^\cdot\otimes\Sigma^{\alpha_1^*}(S_{i_2}/S_{i_1})^*)$ with this resolution. Now, since F is a relative Grassmanian over G, $F=G(i_1,S_{i_2})$, $\mathscr{F}^\cdot$ is a convolution of the following true complex over $D^b(F)$:
$$\dots\to\bigoplus_{|\alpha_1|=i}p^*(Rp_*(\mathscr{F}^\cdot\otimes\Sigma^{\alpha_1^*}(S_{i_2}/S_{i_1})^*))\otimes\Sigma^{\alpha_1}S_{i_1}\to\dots\tag{3.2}$$
which is equipped with a canonical Postnikov system, realising $\mathscr{F}^\cdot$as a convolution. We can substitute here in the place of each $Rp_*(\dots)$ its resolution. Due to the vanishing of higher Ext's between elements of X, we can then form the convolution of (3.2), using only genuine homotopy classes of morphisms of complexes. This yields the desired resolution. □

The resolution thus obtained has terms, which can be explicitly expressed through $\mathscr{F}^\cdot$. Denote for short α a whole sequence $(\alpha_1,\dots,\alpha_k)$ of Young diagrams, and set
$$|\alpha|=\sum|\alpha_j|,\qquad E_\alpha=\bigotimes_{j=1}^k\Sigma^{\alpha_j}S_{i_j},\qquad \Phi_\alpha=\otimes\Sigma^{\alpha_j^*}(S_{i_{j+1}}/S_{i_j}),$$
where $S_{i_{k+1}}=V$. In this notation the p-th term of the resolution has the form
$$\bigoplus_{i-j=p}\ \bigoplus_{|\alpha|=j}\mathbb{H}^i(F,\mathscr{F}^\cdot\otimes\Phi_\alpha)\otimes E_\alpha.\tag{3.3}$$
One cannot construct an analogous resolution by means of Φ_α instead of E_α, since Φ_α do have higher Ext's between themselves. We shall construct below a spectral sequence with the first term $\mathbb{H}^\cdot(F,\mathscr{F}^\cdot\otimes E_\alpha)\otimes\Phi_\alpha$ converging to $\underline{H}^\cdot(\mathscr{F}^\cdot)$.

3.11. Example. A resolution of the structure sheaf of a point on F can be obtained as a Koszul resolution. Choose linear subspaces V^{i_k}, $V^{n-i_k+i_{k-1}}, \ldots, V^{n-i_2+i_1}$ in V, $\dim V^j = j$, which are in general position. Then the subspaces

$$V^{i_k}\cap V^{n-i_k+i_{k-1}}\cap\ldots\cap V^{n-i_2+i_1}, \qquad V^{i_k}\cap\ldots\cap V^{n-i_3+i_2}, \ldots, V^{i_k}$$

form a flag of type $(i_1, \ldots, i_k)$, i.e. a point of F. Consider the bundle

$$(S_{i_1}^*\otimes V/V^{n-i_2+i_1})\oplus(S_{i_2}^*\otimes V/V^{n-i_3+i_2})\oplus\ldots\oplus(S_{i_k}^*\otimes V/V^{i_k})$$

over F and its section $\sigma=(\sigma_1, \ldots, \sigma_k)$, where

$$\sigma_j\in H^0(F, S_{i_j}^*\otimes V/V^{n-i_{j+1}+i_j}) = \operatorname{Hom}(V, V/V^{n-i_{j+1}+i_j})$$

is the natural projection. This section vanishes exactly in the described point and generates the Koszul resolution. Due to the Cauchy formula terms of this resolutions are direct sums of elements of X. In fact, this resolution is a particular case of (3.3). One cannot, however, construct a resolution of the diagonal in $G\times G$ in this way, since it is impossible to choose a family of subspaces with desired properties, which depend analytically on the point $x\in F$.

3.12. Let us introduce on the set $X=X(i_1, \ldots, i_k, n)$ a binary relation $\trianglerighteq$, setting $E\trianglerighteq F$ iff there are bundles $E_1, \ldots, E_m\in X$ such that $\operatorname{Hom}(E, E_1)\neq 0, \ldots, \operatorname{Hom}(E_m, E)\neq 0$. We shall identify X with the set of corresponding k-tuples of Young diagrams and write $\alpha\trianglerighteq\beta$ if $E_\alpha\trianglerighteq E_\beta$ and $\alpha\trianglelefteq\beta$ if $\beta\trianglerighteq\alpha$. Set also $l(\alpha)=\sum_{j=1}^{k}(k+1-j)|\alpha_j|$. The notation $y\subset z\otimes u$, where y, z, u, are Young diagrams, will mean that $\Sigma^y\subset\Sigma^z\otimes\Sigma^u$. Considering the direct images of the sheaf

$$\underline{\operatorname{Hom}}(E_\beta, E_\alpha) = \otimes\underline{\operatorname{Hom}}(\Sigma^{\beta_j}S_{i_j}, \Sigma^{\alpha_j}S_{i_j})$$

in the projections

$$F(i_1, \ldots, i_k, V)\to F(i_2, \ldots, i_k, V)\to\ldots\to G(i_k, V),$$

we obtain the following criterion for the non-vanishing of $\underline{\operatorname{Hom}}(E_\beta, E_\alpha)$:

There are Young diagrams $x_1, x_2, \ldots$ such that x_1 has $\leqq i_1$ rows, x_{2j} and x_{2j+1} have $\leqq i_j$ rows $(j\geqq 1)$ and $x_1\otimes\alpha_1\supset\beta_1$, $x_1\otimes\beta_2\supset x_2$, $x_3\otimes\alpha_2\supset x_2$, $x_3\otimes\beta_3\supset x_4$, $x_5\otimes\alpha_3\supset x_4$ etc. (*)

It follows from this criterion that $\trianglelefteq$ is a partial ordering. Indeed, we must only verify the property ($\alpha\trianglelefteq\beta$ and $\beta\trianglelefteq\alpha\Rightarrow\alpha=\beta$). To do this, we prove that if $\alpha\triangleleft\beta$ (i.e. $\alpha\trianglelefteq\beta$ and $\alpha\neq\beta$), then $l(\alpha)\leqq l(\beta)$. We can assume that $\operatorname{Hom}(E_\beta, E_\alpha)\neq 0$. Let x_1 be the Young diagrams, given by (*). Then

$$\alpha_1\leqq\beta_1, \quad |x_1|=|\beta_1|-|\alpha_1|, \quad |x_2|=|\beta_2|+|\beta_1|-|\alpha_1|, \quad x_2\geqq\alpha_2.$$

Hence $|\beta_2|+|\beta_1|\geqq|\alpha_2|+|\alpha_1|$. Thus continuing, we obtain for each ν the inequality $\sum_{j=1}^{\nu}|\beta_j|\geqq\sum_{j=1}^{\nu}|\alpha_j|$. Summing them up, we obtain that $l(\beta)\geqq l(\alpha)$. If $\beta\neq\alpha$, then at least one of these inequalities in strict, hence $l(\beta)>l(\alpha)$.

So, $\trianglelefteq$ is a partial ordering and l is a monotonous function. Also, it is clear that $\operatorname{Hom}(E_\alpha, E_\alpha)=\mathbb{C}$. Hence, $\mathfrak{A}=\mathfrak{A}(i_1, \ldots, i_k, n)$ is an ordered in the sense of §2 pre-additive category. The complex $B^\bullet_{\mathfrak{A}}$ from n.2.16 is a resolution of the diagonal on $F\times F$. However, this resolution is very cumbersome. The aim of the remainder of this section is to construct a smaller "resolution" (3.4) and corresponding generalised Beilinson spectral sequences.

3.13. Proposition. *The poset X is ranked with rank function l.*

Proof. Let D be the set of k-tuples $\alpha=(\alpha_1, \ldots, \alpha_k)$ of Young diagrams such that α_j has $\leqq i_j$ rows. For $\alpha\in D$ set $E_\alpha=\bigotimes_{j=1}^{k}\Sigma^{\alpha_j}S_{i_j}$ and introduce a relation $\trianglelefteq$ on D similarly to n.3.12. The criterion $(*)$, the fact that $\trianglelefteq$ is a partial order and the monotoneity of the function $l(\alpha)=\sum_{j=1}^{k}(k+1-j)|\alpha_j|$ remain in force. The inclusion $X\subset D$ preserves the order.

3.14. Lemma. *D is a ranked poset with rank function l.*

Proof. It is enough to show that if $\alpha\lhd\beta$ is an elementary pair (i.e. there is no γ such that $\alpha\lhd\gamma\lhd\beta$), than $l(\beta)=l(\alpha)+1$. This statement is obvious for $k=1$. Consider next the case $k=2$. Let $(\alpha_1, \alpha_2)\lhd(\beta_1, \beta_2)$ be an elementary pair , x_1, x_2-Young diagrams such that $x_1\otimes\alpha_1\supset\beta_1$ (in particular, $\alpha_1\leqq\beta_1$), $x_1\otimes\beta_2\supset x_2\geqq\alpha_2$. Then we have $(\alpha_1, \alpha_2)\trianglelefteq(\alpha_1, x_2)\trianglelefteq(\beta_1, \beta_2)$. Since $(\alpha_1, \alpha_2)\lhd(\beta_1, \beta_2)$ is an elementary pair, one of these non-strict inequalities is in fact an equality. If $\alpha_1=\beta_1$, then the fact that $l(\beta)=l(\alpha)+1$ is obvious. Hence we suppose that $\alpha_2=x_2$. So, $x_1\otimes\alpha_1\supset\beta_1$, $x_1\otimes\beta_2\supset\alpha_2$. In particular,

$$|\alpha_1|+|\alpha_2|=|\beta_1|+|\beta_2| \quad\text{and}\quad l(\beta)-l(\alpha)=|\beta_1|-|\alpha_1|=|x_1|.$$

Suppose that $|x_1|>1$. Then x_1 has more than one row or more than one column. Consider first the possibility that it has more than one row. Write in cells of x_1 numbers according to the Littlewood-Richardson rule 1.9: in the cells of the first row we write numbers 1, in cells of the second row-number 2 etc. The diagram β_1 is obtained from α_1 and α_2 – from x_2 by admissible (i.e. satisfying the conditions 1)–3) of n.1.9) adjoining of cells of x_1. Let y denote the diagram x_1 with its last row deleted, and z-this last row; denote v the number of this row. Deleting from β_1 all cells with numbers v, we obtain a Young diagram γ_1 such that $y\otimes\alpha_1\supset\gamma_1$ and $z\otimes\gamma_1\supset\beta_1$. Since the tensor multiplication of representations is commutative up to isomorphism, α_2 can be obtained from β_2 by first adjoining the cells of z and that by admissible adjoining the cells of y. Denote γ_2 the intermediate diagram: $\beta_2\otimes z\supset\gamma_2$, $\gamma_2\otimes y\supset\alpha_2$. Then we have $(\alpha_1, \alpha_2)\lhd(\gamma_1, \gamma_2)\lhd(\beta_1, \beta_2)$. This contradiction shows that x_1 cannot have more than one row. The use of the dual Littlewood-Richardson rule shows that x_1 cannot have more than one column, and our statement is proved for $k=2$.

For $k>2$ the statement is proved by induction. Suppose that $(\alpha_1, \ldots, \alpha_k)\lhd(\beta_1, \ldots, \beta_k)$ is an elementary pair. The criterion $(*)$ gives Young diagrams $x_1, \dots, x_{2k-2}$ such that

$$x_1 \otimes \alpha_1 \supset \beta_1, \quad x_1 \otimes \beta_2 \supset x_2, \dots, x_{2k-3} \otimes \alpha_{k-1} \supset x_{2k-4},$$
$$x_{2k-3} \otimes \beta_k \supset x_{2k-2} \geqq \alpha_k.$$

From these relations we have

$$(\alpha_1, \dots, \alpha_k) \trianglelefteq (\alpha_1, \dots, \alpha_{k-1}, x_{2k-2}) \trianglelefteq (\beta_1, \dots, \beta_k).$$

If the right non-strict inequality is in fact an equality, then the statement is obvious. Suppose therefore that $\alpha_k = x_{2k-2}$. This entails the equality

$$\sum_{j=1}^{k} |\alpha_j| = \sum_{j=1}^{k} |\beta_j|.$$

Also, we have

$$(\alpha_1, \dots, \alpha_k) \trianglelefteq (\alpha_1, \dots, \alpha_{k-2}, x_{2k-4}, \beta_k) \trianglelefteq (\beta_1, \dots, \beta_k).$$

Hence, either $(\alpha_{k-1}, \alpha_k) \trianglelefteq (x_{2k-4}, \beta_k)$ is an elementary pair of sequences of length 2, or $(\alpha_1, \dots, \alpha_{k-2}, x_{2k-4}) \triangleleft (\beta_1, \dots, \beta_{k-1})$ is an elementary pair of sequences of length $k-1$. In both cases the statement follows from the induction hypothesis. Lemma 3.14 is proved.

If $\alpha \triangleleft \beta$ is an elementary pair in D, we shall say that α is obtained from β by modification. Lemma 3.14 tells us that one can take from β_j one cell in such way that the rest were a Young diagram; then, this cell is to be added to β_{j+1} in such way that the result were a Young diagram (in the case $j=k$ the taken cell is to be thrown away). We can consider this as a "game", which takes place on k "boards", the j-th boards has the form of a rectangular semi-strip, infinite to the right and having the width i_j (the cells are thought of as squares 1×1). One can play also on finite boards. Suppose given finite rectangular boards T_j, $j=1, \dots, k$, of integer sizes and let T_j^+ be the boards of the form of rectangular semistrips, obtained from T_j by infinite continuation to the right. We shall call a position any arrangement of cells on a given set of boards, which on every board forms a Young diagram in its left upper corner. Proposition 3.12 is now a consequence of the following lemma.

3.15. Lemma. *Let α, β be two positions on a set of finite rectangular boards T_j, $j=1, \dots, k$. If α can be obtained from β by means of a set of modifications on the set of extended boards T_j^+, then α can be obtained from β by means of (possibly) other set of modifications, which display themselves within the boards T_j.*

Proof. Consider the poset of positions as a category Π. Modifications are then morphisms of this category. We shall say that a modification has type j, if a cell is taken from the j-th board. Let m_j, m_l be modifications of types j, l and the composition $m_l m_j$ is defined. It is immediate to see that there are modifications m_l', m_j' of types l, j such that $m_l m_j = m_j' m_l'$. Hence each composition of modifications equals in Mor Π to a morphism of the form $M_1 \dots M_k$, where M_j is some composition of modifications of type j. Now, if α is obtained from

β by means of such a sequence of modifications (on the set of extended boards T_j^+), then all intermediate positions lie in fact within T_j. The lemma and with it the Proposition 3.13, are proved.

3.16. Since $X = \mathrm{Ob}\,\mathfrak{A}$ is a ranked poset, we can apply the remark of n.2.18, obtaining the following true complex over $D^b(F\times F)$:

$$\ldots \to \bigoplus_{l(\alpha)=i} E_\alpha \boxtimes \hat{E}_\alpha[i] \to \ldots \tag{3.4}$$

and a canonical Postnikov system, realising $B_{\mathfrak{A}}$, hence $\mathcal{O}_\Delta$, as a convolution of this complex. Observe, that the complex $\hat{E}_\alpha$ has only one non-trivial cohomology sheaf, in degree $-|\alpha|$. Indeed, by homogeneity, the sheaves $\underline{H}^j(\hat{E}_\alpha)$ are locally free. Their fibres (bundle-theoretical) over some point x equal to $\mathbb{H}^j(F, \hat{E}_\alpha \otimes P^\cdot)$, where $P^\cdot$ is any locally free resolution of $\mathcal{O}_x$. Using the Koszul resolution (Example 3.10) and the orthogonality relations 2.15, we obtain the claimed fact.

Below we shall show that $\underline{H}^{-|\alpha|}(\hat{E}_\alpha) = \Phi_\alpha$. Meanwhile denote this cohomology sheaf by Φ'_α. The complex (3.4) can thus be rewritten as

$$\ldots \to \bigoplus_{l(\alpha)=i} E_\alpha \boxtimes \Phi'_\alpha[i-|\alpha|] \to . \tag{3.5}$$

It generates generalised Beilinson spectral sequences converging to $\underline{H}^\cdot(\mathcal{F}^\cdot)$ for $\mathcal{F} \in \mathrm{Ob}\, D^b(F)$:

$$E_1^{pq} = \bigoplus_{l(\alpha)=-p} \mathbb{H}^{q-p+|\alpha|}(F, \mathcal{F}^\cdot \otimes \Phi'_\alpha) \otimes E_\alpha,$$

$$'E_1^{pq} = \oplus \mathbb{H}^{q-p+|\alpha|}(F, \mathcal{F}^\cdot \otimes E_\alpha) \otimes \Phi'_\alpha.$$

To obtain, say, the first one, note that the argument with pull-back of $\mathcal{F}^\cdot$ to $F\times F$, tensoring with (3.5) and then pushdown in the other direction yields a true complex over $D^b(F)$:

$$\ldots \to \bigoplus_{l(\alpha)=i} R\Gamma(F, \mathcal{F}^\cdot \otimes \Phi'_\alpha)[i-|\alpha|] \otimes E_\alpha \to \ldots$$

and a canonical Postnikov system, realising $\mathcal{F}^\cdot$ as its convolution. The desired spectral sequence follows now from n.1.3.

3.17. Lemma. $\Phi'_\alpha = \Phi_\alpha$.

Proof. It suffices to show that Φ_α satisfy the orthogonality relations

$$H^i(F, E_\alpha \otimes \Phi_\beta) = \begin{cases} 0 & \text{for all } i, & \text{if } \alpha \neq \beta \\ 0 & \text{for } i \neq |\alpha|, & \text{if } \alpha = \beta \\ \mathbb{C} & \text{for } i = |\alpha|, & \text{if } \alpha = \beta \end{cases}$$

(then the statement will follow from the spectral sequence $'E$). But these relations are easy consequences of Lemma 3.2b). □

Let us restate our result in its final form.

3.18. Proposition. *For each $\mathscr{F} \in \mathrm{Ob}\, D^b(F)$ there are natural spectral sequences*

$$E_1^{pq} = \bigoplus_{l(\alpha)=p} \mathbb{H}^{q-p+|\alpha|}(F, \mathscr{F} \otimes \Phi_\alpha) \otimes E_\alpha,$$

$$'E_1^{pq} = \oplus \mathbb{H}^{q-p+|\alpha|}(F, \mathscr{F} \otimes E_\alpha) \otimes \Phi_\alpha$$

converging to $\underline{H}^{p+q}(\mathscr{F})$. Here α runs cover k-tuples $(\alpha_1, \ldots, \alpha_k)$ of Young diagrams,

$$E_\alpha = \bigotimes_{j=1}^{k} \Sigma^{\alpha_j}(S_{i_j}), \qquad \Phi_\alpha = \otimes \Sigma^{\alpha_j^*}(S_{i_{j+1}}/S_{i_j})^*.$$

§ 4. The case of quadrics

4.1. Let E be an N-dimensional vector space ($N \geqq 3$), endowed with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$, $Q = Q(E) \subset P(E)$ – the quadric of isotropic 1-dimensional subspaces in E, $B = \oplus H^0(Q, \mathscr{O}(i))$ – the projective coordinate algebra of Q. Define the graded Clifford algebra $A = A(E) = \oplus A_i(E)$ to be generated by $\xi \in E$, $\deg \xi = 1$, and by a letter h, $\deg h = 2$, which are subject to the relations $\xi\eta + \eta\xi = 2\langle\xi, \eta\rangle h$, $\xi h = h\xi$ for all $\xi, \eta \in E$. The algebras A and B are Koszul quadratic algebras, Priddy dual to each other, i.e. $A = \mathrm{Ext}_B^{\cdot}(\mathbb{C}, \mathbb{C})$, $B = \mathrm{Ext}_A^{\cdot}(\mathbb{C}, \mathbb{C})$. The corresponding generalised Koszul complex

$$\ldots \to A_2^* \otimes B \to A_1^* \otimes B \to A_0^* \otimes B \tag{4.1}$$

coincides with the Tate resolution [20] of B-module $\mathbb{C}$. The differential in (4.1) equals to $\sum r_{\xi_i}^* \otimes l_{x_i}$ where $\xi_i \in E$, $x_i \in E^*$ are dual bases, $l_x \colon B \to B$ for $x \in E^* = B_1$ sends b to xb, $r_\xi \colon A \to A$ for $\xi \in E = A_1$ sends a to $a\xi$.

If $\xi \in E$ is an isotropic vector, then $\xi^2 = 0$ in A. If $M = \oplus M_i$ is a graded left A-module, then the sequence of sheaves on Q

$$\mathscr{L}(M) = \{\ldots \to \tilde{M}_{-1}(-1) \to \tilde{M}_0 \to \tilde{M}_1(1) \to \ldots\}$$

is a complex (see also [5]).

Consider $A^* = \oplus A_i^*$ with grading $\deg A_i^* = -i$, equipped with the obvious structure of left A-module. The complex $\mathscr{L}(A^*)$ is exact, since the corresponding (by Serre) complex of graded B-modules is just the Tate resolution (4.1). The complex $\mathscr{L}(A)$ is also exact, since it is dual to $\mathscr{L}(A^*)$, which is (due to homogeneity) fiberwise exact as a complex of vector bundles.

Set $\Psi_i = \mathrm{Ker}\{\tilde{A}_i^* \to \tilde{A}_{i-1}^*(1)\}$, $i \geqq 0$. It is a locally free sheaf, which has a right resolution $\{\tilde{A}_i^* \to \tilde{A}_{i-1}^*(1) \to \ldots \to \tilde{A}_0^*(i)\}$.

If W is an orthogonal vector bundle over a base variety Y, then we define, as above, a sheaf of graded algebras $A(W) = \oplus A_i(W)$ over Y; $A_i(W)$ is a vector bundle, isomorphic to $\bigoplus_{p \geqq 0} \Lambda^{i-2p}(W)$; the corresponding isomorphism sends $\Lambda^{i-2p}(W)$ first to $A_{i-2p}(W)$ by means of antisymmetrisation of product, and then multiplies the result by h^p. Denote $S^\dagger/S$ the orthogonal bundle over Q, whose fiber over an isotropic $l \subset E$, $\dim l = 1$, is $l^\dagger/l$, where $l^\dagger$ is the orthogonal complement.

4.2. Proposition. *There is an exact sequence of bundles over Q:*

$$0 \to A_{i-1}(S^\dagger/S)(-1) \xrightarrow{\alpha} \Psi_i \xrightarrow{\beta} A_i(S^\dagger/S) \to 0$$

Proof. We shall identify $\Lambda^i(E)$ with its image in $A_i(E)$ under the antisymmetrized product map, so that $A_i(E) = \bigoplus_{p \geq 0} \Lambda^{i-2p}(E)\, h^p$. Introduce on A_i a scalar product, setting the summands in this sum to be mutually orthogonal, and introducing on each $\Lambda^{i-2p}(E)$ the scalar product, induced by $\langle\, ,\rangle$ on E. Having thus identified A_i with A_i^*, we obtain for each $\xi \in E$ maps $l_\xi^*, r_\xi^*: A_i \to A_{i-1}$. For each $\xi \in E$ denote $? \lrcorner \xi: \Lambda^\cdot(E) \to \Lambda^\cdot(E)$ the convolution with ξ: for $\eta \in \Lambda^1(E)$ $\eta \lrcorner \xi = \langle \eta, \xi\rangle$; for homogenous ω_1, ω_2 $(\omega_1 \wedge \omega_2) \lrcorner \xi = \omega_1 \wedge (\omega_2 \lrcorner \xi) + (-1)^{\deg \omega_2}(\omega_1 \lrcorner \xi) \wedge \omega_2$. Then for $\omega \in \Lambda^p(E) \subset A_p$, $\xi \in E$ we have (Wick's theorem):

$$\xi(\omega h^k) = (\xi \wedge \omega)\, h^k + (\omega \lrcorner \xi)\, h^{k+1}, \qquad l_\xi^*(\omega h^k) = (\omega \lrcorner \xi)\, h^k + (\omega \wedge \xi)\, h^{k-1}. \tag{4.2}$$

The fiber of Ψ_i over $\mathbb{C}v \in Q$ is $\operatorname{Ker}\{l_v^*: A_i \to A_{i-1}\}$. Let $\bar{\xi}_1, \ldots, \bar{\xi}_{i-1} \in v^\dagger/\mathbb{C}v$, $\xi_j \in v^\dagger$-their arbitrary representatives. Set $\alpha(\bar{\xi}_1 \ldots \bar{\xi}_{i-1}) = v\xi_1 \ldots \xi_{i-1}$. It follows from (4.2) that we obtain a well-defined map $\alpha: A_{i-1}(S^\dagger/S)(-1) \to \Psi_i$. In order to define β, define $\beta^*(-1): A_i(S^\dagger/S) \to \Psi_i^*(-1) = \operatorname{Ker}\{\tilde{A}_{i+1} \to \tilde{A}_{i+2}(1)\}$ by similar formula $\beta^*(-1)(\bar{\xi}_1 \ldots \bar{\xi}_i) = v\xi_1 \ldots \xi_i$. We claim that $\beta\alpha = 0$. To see this, consider $\mathbb{C}v \in Q$, $a \in A_{i-1}(v^\dagger/\mathbb{C}v)$ and suppose that $\alpha(a) = l_v^*(x)$ for some $x \in A_{i+1}$. We must show that $\langle x, \beta^*(b)\rangle = 0$ for all $b \in A_i(v^\dagger/\mathbb{C}v)$. It suffices to consider a and b of the form $a = (\bar{w}_1 \wedge \ldots \wedge \bar{w}_j)\, h^k$, $b = (\bar{w}'_1 \wedge \ldots \wedge \bar{w}'_r)\, h^s$, where $\bar{w}_\nu, \bar{w}'_\nu \in v^\dagger/\mathbb{C}v$ and the expressions inside the brackets are antisymmetrized products. Choose representatives w_ν, w'_ν of $\bar{w}_\nu, \bar{w}'_\nu$. Then $\alpha(a) = (v \wedge w_1 \wedge \ldots \wedge w_j)\, h^k = l_v^*(x)$, where $x = (w_1 \wedge \ldots \wedge w_j)\, h^{k+1}$. The scalar product $\langle x, \beta^*(b)\rangle$ can be non-zero only if $s = k+1$, $r = j-1$. In this case $\langle w_1 \wedge \ldots \wedge w_j, v \wedge w'_1 \wedge \ldots \wedge w'_{j-1}\rangle = 0$, since $w_i \perp v$. So, $\beta\alpha = 0$. It is clear that α is injective and β is surjective. The exactness follows now from dimension count.

4.3. Let $Cl = Cl(E) = A(E)/(h-1)A(E)$ be the usual Clifford algebra of E, with its $\mathbb{Z}/2$-grading: $Cl(E) = Cl_{\bar{0}}(E) \oplus Cl_{\bar{1}}(E)$. If N is even, then the algebra $Cl(E)$ is simple. The irreducible left $Cl(E)$-module M, being restricted to $Cl_{\bar{0}}(E)$, is decomposed canonically into the sum of two irreducible summands: $M = M_+ \oplus M_-$. One can form A-modules $\mathcal{M}_- = \{M_-, M_+, M_-, \ldots\}$, $\mathcal{M}_+ = \{M_+, M_-, M_+, \ldots\}$, in which h acts by identity and the grading starts from zero. The complexes $\mathcal{L}(\mathcal{M}_\pm)$ are right resolutions of some vector bundles over Q. If $N \equiv 0 \pmod 4$, we define the bundles Σ_+ and Σ_- over Q as dual to the bundles, defined by complexes $\mathcal{L}(\mathcal{M}_+)$ and $\mathcal{L}(\mathcal{M}_-)$, respectively. If $N \equiv 2 \pmod 4$, we define Σ_+ and Σ_- in the other way: Σ_+^* is quasi-isomorphic to $\mathcal{L}(\mathcal{M}_-)$ and vice versa. If N is odd, then $Cl(E)$ is a sum of two simple algebras. To an irreducible module M over any of its simmands corresponds A-module $\mathcal{M} = \{M, M, M, \ldots\}$ (the grading starts from zero). This module does not, up to isomorphism, depend on the choice of the summand. The complex $\mathcal{L}(\mathcal{M})$ is a resolution of some bundle over Q, whose dual we denote Σ. In this notations, we have: for N odd, $\Sigma^* = \Sigma(-1)$; for $N \equiv 0 \pmod 4$, $\Sigma_\pm^* = \Sigma_\mp(-1)$; for $N \equiv 2 \pmod 4$, $\Sigma_\pm^* = \Sigma_\pm(-1)$, as follows from the behaviour of spinor representations under dualisation [6].

For $N=3$, $Q=P^1$, the bundle Σ is $\mathcal{O}_{P^1}(1)$. For $N=4$, $Q=P^1\times P^1$, the bundles Σ_+ and Σ_- are $\mathcal{O}_{P^1\times P^1}(1,0)$ and $\mathcal{O}_{P^1\times P^1}(0,1)$. For $N=6$, Q is the Grassmanian $G(2,\mathbb{C}^4)$, the bundles Σ_+ and Σ_- are S^* and $\mathbb{C}^4/S$, where S is the tautological 2-dimensional bundle over the Grassmanian.

The bundles Σ or Σ_+, Σ_- are the spinor bundles for the orthogonal bundle $S^\dagger/S$ over Q. In other words, we have the following fact.

4.4. Proposition. *$Cl(S^\dagger/S)$ is isomorphic, as a sheaf of algebras on Q, to $\underline{\mathrm{End}}(\Sigma_+\oplus\Sigma_-)$ for N even, and to $\underline{\mathrm{End}}(\Sigma)\oplus\underline{\mathrm{End}}(\Sigma)$ for N odd.*

Proof. Let $\mathbb{C}v\in Q$ and $w\perp v$. Then w anticommutes with v in A. Hence, for N even, w defines a transformation $\mathrm{Ker}\{v\colon M_\pm\to M_\mp\}\to\mathrm{Ker}\{v\colon M_\mp\to M_\pm\}$, i.e. a map from the fiber of $\Sigma_\pm$ over $\mathbb{C}v$ to the fiber of $\Sigma_\mp$. This defines the required isomorphism $Cl(S^\dagger/S)\to\underline{\mathrm{End}}(\Sigma_+\oplus\Sigma_-)$. The case of odd N is considered similarly. □

4.5. Consider on $Q\times Q$ the following complex of sheaves $C^\cdot$. Its terms C^{-i} equal $\Psi_i\boxtimes\mathcal{O}(-i)$ (so $C^\cdot$ is infinite to the left). The differential $\Psi_i\boxtimes\mathcal{O}(-i)\to\Psi_{i-1}\boxtimes\mathcal{O}(-i+1)$ arises from the morphism of complexes

$$\begin{array}{c}
\{\tilde{A}_i^* \longrightarrow \tilde{A}_{i-1}^*(1) \longrightarrow \dots\}\ \boxtimes\mathcal{O}(-i)\\
\Big\downarrow{\scriptstyle d}\\
\{\tilde{A}_{i-1}^* \longrightarrow \tilde{A}_{i-2}^*(1) \longrightarrow \dots\}\boxtimes\mathcal{O}(-i+1)
\end{array}$$

which on each term equals $\Sigma l_{\xi_i}^*\otimes l_{x_i}$, in the notation of n.4.1. Since the structure of left A-module on A^* is dual to the structure of right A-module on A, and the multiplication from the left commutes with the multiplication from the right, d is indeed a morphism of complexes.

We are going to show that $C^\cdot$ is a resolution of the sheaf $\mathcal{O}_\Delta$, where $\Delta\subset Q\times Q$ is the diagonal. This fact is valid in more general context: for an arbitrary projective variety $Y\subset P^n$ such that its projective coordinate algebra $\mathbb{C}[Y]$ is Koszul (as shown in [1], there are "very many" such varieties). Until the end of n.4.6, B_Y will denote the algebra $\mathbb{C}[Y]$, $A=\mathrm{Ext}^\cdot_{B_Y}(\mathbb{C},\mathbb{C})$-its Priddy dual. To each graded left A-module corresponds, as above, a complex $\mathscr{L}(M)$ of sheaves on Y, and $\mathscr{L}(A^*)$ is exact. The definitions of sheaves Ψ_i and of the complex $C^\cdot$ also extend literally to this case.

4.6. Proposition. *If $Y\subset P^n$ is a projective variety such that $B_Y=\mathbb{C}[Y]$ is a Koszul algebra, then the complex of sheaves on $Y\times Y$*

$$\{\dots\to\Psi_2\boxtimes\mathcal{O}(-2)\to\Psi_1\boxtimes\mathcal{O}(-1)\to\mathcal{O}_{Y\times Y}\}=C^\cdot$$

is a left resolution of $\mathcal{O}_\Delta$, $\Delta\subset Y\times Y$ being the diagonal.

Proof. We shall write B for B_Y. Set $B^2=\oplus B_i\otimes B_i$, $M(B^2)$ the category of graded B^2-modules, $Qsh(Y\times Y)$ – the category of quasicoherent sheaves on $Y\times Y$, $L\colon M(B^2)\to Qsh(Y\times Y)$ – the localisation functor of Serre [9]. It is exact and compatible with infinite direct sums. Making use of Serre's theorem, which connects coherent sheaves on a projective variety and finitely generated modules over its coordinate algebra, and considering the resolution $\Psi_i \sim \{\tilde{A}_i^* \to \tilde{A}_{i-1}^*(1) \to \ldots \to \tilde{A}_0^*(i)\}$, we find that the complex $C^\cdot$ is quasi-isomorphic to $L(D^\cdot)$, where $D^\cdot$ is the total complex of the following double complex $D^{\cdot\cdot}$ of B^2-modules:

$$\begin{tikzcd}
\ldots \arrow[r] & \bigoplus\limits_i B_{i-2}\otimes B_{i+2} & & \\
\ldots \arrow[r] & \bigoplus\limits_i B_{i-2}\otimes A_1^*\otimes B_{i+1} \arrow[u] \arrow[r] & \bigoplus\limits_i B_{i-1}\otimes B_{i+1} & \\
\ldots \arrow[r] & \bigoplus\limits_i B_{i-2}\otimes A_2^*\otimes B_i \arrow[u] \arrow[r] & \bigoplus\limits_i B_{i-1}\otimes A_1^*\otimes B_i \arrow[u] \arrow[r] & \bigoplus\limits_i B_i\otimes B_i .
\end{tikzcd}$$

There is a natural morphism of complexes $\phi: D^\cdot \to \oplus B_{2i}$, compatible (under the identification $L(\oplus B_{2i}) = \mathcal{O}_\Delta$) with the morphism $C^\cdot \to \mathcal{O}_\Delta$. We want to show that ϕ is a quasi-isomorphism.

The terms of the double complex $D^{\cdot\cdot}$ have themselves a grading. According to this grading, $D^{\cdot\cdot}$ splits into a direct sum of double complexes $D_i^{\cdot\cdot}$, $i\in\mathbb{Z}$. To specify $D_i^{\cdot\cdot}$ exactly, we can define it to be the summand, containing the summand $B_i\otimes B_i$ of the term in the right od $D^{\cdot\cdot}$. Then, each $D_i^{\cdot\cdot}$ has only finitely many non-zero terms and rows of $D_i^{\cdot\cdot}$ are just graded parts of the generalised Koszul complex

$$\{B_0\otimes A_r^* \to B_1\otimes A_{r-1}^* \to \ldots \to B_r\otimes A_0^*\}$$

tensored with some vector spaces. Hence, they all are exact except those, which contain only one non-zero term, i.e. $B_0\otimes A_0^*\otimes B_{2i}$ (for $D_i^{\cdot\cdot}$). This term maps identically by ϕ to B_{2i}. So, ϕ is a quasi-isomorphism. The proposition is proved.

We return now to the case of quadrics.

4.7. Proposition. *The kernel of the differential $C^{-N+3} \to C^{-N+4}$ is isomorphic to $\Sigma(-1)\boxtimes\Sigma(-N+2)$ for N odd, to*

$$\Sigma_+(-1)\boxtimes\Sigma_+(-N+2)\oplus\Sigma_-(-1)\boxtimes\Sigma_-(-N+2)\ \textit{for}\ N\equiv 2 \ (\mathrm{mod}\ 4),$$

and to

$$\Sigma_+(-1)\boxtimes\Sigma_-(-N+2)\oplus\Sigma_-(-1)\boxtimes\Sigma_+(-N+2)\ \textit{for}\ N\equiv 0 (\mathrm{mod}\ 4).$$

Proof. The multiplication by h induces isomorphisms $A_i \to A_{i+2}$ for $i \geqq N-1$, and the projection $A_i \to Cl_{\bar{i}}(E)$ is an isomorphism for such i. Here and below $\bar{i}$ means i modulo 2. Correspondingly, we have $\Psi_i \simeq \Psi_{i+2}$, $i \geqq N-2$, $C^{i-2} \simeq C^i(0,-2)$, $i \leqq -N+2$. Denote the kernel in question by R. Then R is the cokernel of the "stable" differential $C^{-N+1} \to C^{-N+2}$. Hence, one can construct a left resolution of R as the total complex of the following double complex:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \widetilde{Cl}^*_{\bar{N}+\bar{1}}(-2,-N+1) \arrow[r] \arrow[d] & \widetilde{Cl}^*_{\bar{N}}(-2,-N+2) \arrow[d] \\
\cdots \arrow[r] & \widetilde{Cl}^*_{\bar{N}}(-1,-N+1) \arrow[r] & \widetilde{Cl}^*_{\bar{N}-\bar{1}}(-1,-N+2)
\end{tikzcd}
\tag{4.3}
$$

The differentials in (4.3) are maps, dual to the operators of left or right Clifford multiplications by isotropic vectors in E. From the structure of $Cl(E)$ as a bimodule over itself, we find that for odd N the complex (4.3) is isomorphic to $\mathscr{L}(\mathscr{M})^*(-1)\boxtimes\mathscr{L}(\mathscr{M})^*(-N+2)$, for $N\equiv 2 \pmod 4$ – to

$$\mathscr{L}(\mathscr{M})^*(-1)\boxtimes\mathscr{L}(\mathscr{M})^*(-N+2)\oplus\mathscr{L}(\mathscr{M}_+)^*(-1)\boxtimes\mathscr{L}(\mathscr{M}_+)^*(-N+2),$$

and for $N\equiv 0 \pmod 4$ – to

$$\mathscr{L}(\mathscr{M})^*(-1)\boxtimes\mathscr{L}(\mathscr{M}_+)^*(-N+2)\oplus\mathscr{L}(\mathscr{M}_+)^*(-1)$$
$$\boxtimes\mathscr{L}(\mathscr{M}_-)^*(-N+2). \quad \text{Q.E.D.}$$

4.8. We shall use the unified notation $\Sigma_{(\pm)}$ meaning that for even N both bundles Σ_+, Σ_- are considered, and for odd N – the bundle Σ. Consider the following sets of sheaves on Q:

$$X=\{\Sigma_{(\pm)}(-1),\ \Psi_{N-3},\ \ldots,\ \Psi_0=\mathcal{O}\} \qquad Y=\{\Sigma_{(\pm)}(-N+2),\ \mathcal{O}(-N+3),\ \ldots,\ \mathcal{O}\},$$

endowed with indicated here partial orders (for N even, Σ_+ and Σ_- are incomparable; the spinor bundle(s) is (are) less than all others).

4.9. Proposition. *For each two bundles $\mathscr{F}$, $\mathscr{G}$ of the set X* (resp. Y), $\operatorname{Ext}^i(\mathscr{F},\mathscr{G})=0$ *for all* $i>0$, $\operatorname{Hom}(\mathscr{F},\mathscr{G})\neq 0$ *only if* $\mathscr{F}\leq\mathscr{G}$ *and* $\operatorname{Hom}(\mathscr{F},\mathscr{F})=\mathbb{C}$.

Proof. First consider the set Y. Only $\operatorname{Ext}^\bullet(\mathscr{F},\mathscr{G})$, where $\mathscr{F}$ or $\mathscr{G}$ have the form $\Sigma_{(\pm)}(-N+2)$, demand calculation. We proceed as recalled in the end of § 1. Let F be the space of complete isotropic flags in E (one component of such space for even N). It is the quotient of $\mathrm{Spin}(N)$ by the Borel subgroup. Integer weights, i.e. characters of maximal torus in $\mathrm{Spin}(N)$, are determined by n-tuples of numbers $(a_1,\ldots,a_n)$, $n=[N/2]$, which are either all integers, or all half-integers, but not integers. Corresponding invertible sheaf on F will be denoted $\mathcal{O}(a_1,\ldots,a_n)$. Denote also $p\colon F\to Q$ the projection. Then $\Sigma_{(\pm)}=R^0p_*\mathcal{O}((1/2),(1/2),\ldots,(\pm)(1/2))$. The calculation of $\operatorname{Ext}^\bullet(\mathcal{O}(-j),\Sigma_{(\pm)}(-N+2)$ is reduced therefore to the calculation of the cohomology of the invertible sheaf $\mathcal{O}((1/2)+j-N+2,(1/2),\ldots,(\pm)(1/2))$ on F, where $j=0,1,\ldots,N-3$. It is done by using Bott's theorem [13]. Consider, for example, the case of even $N=2n$.

The half-sum ρ of positive roots is $(n-1, n-2, \ldots, 1, 0)$. A weight $(a_1, \ldots, a_n)$ is dominant, iff $a_1 \geqq \ldots \geqq a_{n-1} \geqq |a_n|$. The Weyl group W acts on weights by permutations of components and changes of signs in an even number of places. Adding ρ to the weight corresponding to the invertible sheaf in question, we obtain the weight

$$\chi = (-n+(3/2)+j, n-(3/2), n-(5/2), \ldots, \pm(1/2)).$$

Since $|-n+(3/2)+j| \leqq n-(3/2)$, there are two components of χ having the same absolute value. This property will remain in force after action of an element $w \in W$, sending χ to the dominant Weyl chamber. When after that we shall subtract ρ, we shall obtain a non-dominant weight. This, by Bott's theorem, signifies that $\mathcal{O}((1/2)+j-N+2, (1/2), \ldots, \pm(1/2))$ and hence $\operatorname{Hom}(\mathcal{O}(-j), \Sigma_\pm(-N+2)$ has no cohomology. The sheaves $\underline{\operatorname{Hom}}(\Sigma_{(\pm)}(-N+2), \mathcal{O}(-j))$ for $j=0, 1, \ldots, N-3$ are $R^0 p_*$ of ample invertible sheaves on F and hence have no higher cohomology. To end the proof for the set Y, it remains to calculate Ext's between $\Sigma_{(\pm)}$. Suppose N to be, say, even: $N=2n$. Then $\underline{\operatorname{End}}(\Sigma_+) \oplus \underline{\operatorname{End}}(\Sigma_-) = Cl_{\bar{0}}(S^\dagger/S)$ is isomorphic as a bundle to the sum $\oplus \Lambda^{2i}(S^\dagger/S)$, and $\underline{\operatorname{Hom}}(\Sigma_+, \Sigma_-) \oplus \underline{\operatorname{Hom}}(\Sigma_-, \Sigma_+)$ – to $\oplus \Lambda^{2i+1}(S^\dagger/S)$. The Hodge operator $*$ induces an isomorphism of vector bundles $\Lambda^j(S^\dagger/S) \to \Lambda^{N-2-j}(S^\dagger/S)$, and for $j \leqq n-1$ we have that $\Lambda^j(S^\dagger/S) = R^0 p_* \mathcal{O}(0, 1, \ldots, 1, 0, \ldots, 0)$ (j units). The details, as well as the case of odd n, are left to the reader.

Turn now to the set X. Tensoring the resolutions of sheaves Ψ_i^* and Ψ_j, we find that $\underline{\operatorname{Hom}}(\Psi_i, \Psi_j)$ is quasi-isomorphic to the total complex of the following double complex:

$$\begin{array}{ccccc}
\tilde{A}_i \otimes \tilde{A}_j^* & \longrightarrow & \tilde{A}_i \otimes \tilde{A}_{j-1}^*(1) & \longrightarrow \ldots \longrightarrow & \tilde{A}_i \otimes \tilde{A}_0^*(j) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\tilde{A}_0 \otimes \tilde{A}_j^*(-i) & \longrightarrow & \tilde{A}_0 \otimes \tilde{A}_{j-1}^*(-i+1) & \longrightarrow \ldots \longrightarrow & \tilde{A}_0 \otimes \tilde{A}_0^*(-i+j)
\end{array}$$

graded so that $\deg(\tilde{A}_i \otimes \tilde{A}_j^*) = (0, 0)$. Denote this double complex $R^{\cdot\cdot}$ and its total complex $R^\cdot$. Considering the rows of $H^0(Q, R^{\cdot\cdot})$, we find that $H^0(Q, R^\cdot)$ is quasi-isomorphic to A_{i-j} in the 0-th place (and to zero, if $i-j<0$). Consider first the case $0 \leqq i, j \leqq N-3$. Then the terms of $R^{\cdot\cdot}$ do not have higher cohomology and we deduce that $\operatorname{Ext}^p(\Psi_i, \Psi_j) = 0$, $p>0$, and $\operatorname{Hom}(\Psi_i, \Psi_j) = 0$ unless $i \geqq j$. To calculate Ext's between $\Sigma_{(\pm)}(-1)$ and Ψ_i, $0 \leqq i \leqq N-3$, note that for N even and $i \geqq N-2$ the bundle Ψ_i is a sum of several copies of Σ_+^* or Σ_-^* (depending on the parity of i), and for N even and $i \geqq N-2$, the bundle Ψ_i is a sum of several copies of Σ^*. By scrutinizing the double complex $R^{\cdot\cdot}$ once again we find that $\operatorname{Ext}^p(\Psi_i, \Psi_j) = 0$ for $p>0$ and all i, j. Indeed, Terms of $R^{\cdot\cdot}$ can, in general, have also H^{N-2}. But terms with non-trivial H^{N-2} are situated in such places, that they can contribute only to $\mathbb{H}^p(Q, R^\cdot) = \operatorname{Ext}^p(\Psi_i, \Psi_j)$ for $p<0$. (of course, contributions to negative Ext's must eventually vanish in the spectral sequence). Also, from the form of $R^{\cdot\cdot}$ we find that $\operatorname{Ext}^p(\Psi_i, \Psi_j) = 0$ for all p, if $0 \leqq i \leqq N-3$ and $j \geqq N-2$. The proposition is proved.

Let $\mathfrak{A} = \mathfrak{A}(E)$ and $\mathfrak{B} = \mathfrak{B}(E)$ be the full subcategories in $\operatorname{Sh} Q(E)$ with the sets of objects, respectively, X and Y.

4.10. Theorem. *$D^b(Q(E))$ is equivalent to* $\operatorname{Tr}\mathfrak{A}(E)$ *and to* $\operatorname{Tr}\mathfrak{B}(E)$ *as a triangulated category.*

The theorem follows in a standard way from Propositions 4.6, 4.7 *and* 4.9. □

4.11. Proposition. *Consider the correspondence between sheaves $\mathcal{F}\in Y$ and $\mathcal{G}\in X$ defined by the property: $H^j(Q, \mathcal{F}\otimes G)\neq 0$ for at least one j. This correspondence defines a bijection $\alpha: Y\to X$, sending $\mathcal{O}(-j)\to\Psi_j$, $0\leqq j\leqq N-3$, $\Sigma(-N+2)\to\Sigma(-1)$ for N odd, $\Sigma_\pm(-N+2)\to\Sigma_\mp(-1)$ for $N\equiv 0 \pmod 4$, $\Sigma_\pm(-N+2)\to\Sigma_\pm(-1)$ for $N\equiv 2 \pmod 4$. Sheaves $\mathcal{F}\otimes\alpha(\mathcal{F})$ for $\mathcal{F}\in Y$ have only one nontrivial cohomology group, and this group is isomorphic to $\mathbb{C}$. Its number equals j, if $\mathcal{F}=\mathcal{O}(-j)$, and $N-2$, if $\mathcal{F}=\Sigma_{(\pm)}(-N+2)$.*

The proof, similar to that of proposition 4.9, is left to the reader. □

4.12. The posets X and Y are clearly ranked. Let l be the rank function on Y such that $l(\mathcal{O})=0$, and hence $H^{-l(\mathcal{F})}(Q, \mathcal{F}\otimes\alpha(\mathcal{F}))=\mathbb{C}$, $\mathcal{F}\in Y$. From the generalized Beilinson spectral sequence corresponding to the resolution of the diagonal, constructed in propositions 4.6 and 4.7, we deduce that for $\mathcal{F}\in Y$ the complex $\hat{\mathcal{F}}$, defined according to §2, is quasi-isomorphic to $\alpha(\mathcal{F})[-l(\mathcal{F})]$. Denote $\mathfrak{A}_g$ and $\mathfrak{B}_g$ the graded categories, obtained from $\mathfrak{A}$ and $\mathfrak{B}$ by assigning to $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ the degree $l(\mathcal{F})-l(\mathcal{G})$. Then $\hat{\mathfrak{A}}$ is quasi-isomorphic to $\mathfrak{B}_g$, and $\hat{\mathfrak{B}}$ – to $\mathfrak{A}_g$. In other words, we have

4.13. Corollary. *$\mathfrak{A}_g$ and $\mathfrak{B}_g$ are Koszul categories with quadratic relations, Priddy dual to each other.* □

The finite resolution of the diagonal, construct as in this section, is also a particular case of the complex from 2.18.

4.14. In the rest of this section we describe a modification of the previous construction to the case of incidence quadrics in $P^{N-1}\times\hat{P}^{N-1}$, which is also the flag variety $F(1, N-1, \mathbb{C}^N)$. This modification allows us to give another, different from given in § 3, description of $D^b(F(1, N-1, \mathbb{C}^N))$. The necessary proofs are similar to the case of projective quadrics and are therefore omitted.

So, let E be an N-dimensional vector space, $N\geqq 2$, $L\subset P(E)\times P(E^*)$ – the incidence quadric, defined by the equation $\sum x_i y_i=0$, where $x_i\in E^*$, $y_i\in E$ are dual bases. The bigraded Clifford algebra A is generated by $\xi\in E^*$, $\deg\xi=(1, 0)$, $\eta\in E$, $\deg\eta=(0, 1)$, and h, $\deg h=(1, 1)$, which are subject to relations $\xi\xi'+\xi'\xi=0$, $\eta\eta'+\eta'\eta=0$, $\zeta\eta+\eta\xi=\langle\xi, \eta\rangle h$, $\xi h=h\xi$, $\eta h=h\eta$. Here $\xi, \xi'\in E$, $\eta, \eta'\in E^*$ are arbitrary.

A bigraded left A-module $N=\oplus N_{ij}$ defines a double complex $\mathcal{L}(N)$ with terms $\mathcal{L}(N)^{ij}=N_{ij}\otimes\mathcal{O}(i, j)$.

4.15. Examples. a) Truncations of the module A^*.

$$A^*_{\leqq i, j}=\begin{array}{|cc:}
\cdots & \cdots \\
A^*_{i, j-1} & A^*_{i-1, j-1} \\
A^*_{ij} & A^*_{i-1, j} \\
\hline
\end{array}$$

$\deg A^*_{i,j}=0$. The total complex of the double complex $\mathscr{L}(A^*_{\leqq i,j})$ is a right resolution of a vector bundle on L, denoted Ψ_{ij}.

b) Modules of superfunctions.

$$\mathscr{M}_i=\begin{array}{|cc:}\hdashline \Lambda^{i-1}E & \Lambda^i E \\ \Lambda^i E & \Lambda^{i+1}E \\ \hline\end{array}$$

Generators $\xi\in E$ act on $\mathscr{M}_i$ by Grassmann multiplication, and $\eta\in E^*$ act as Grassmann derivations. If we identify L with $F(1, N-1, E)$ and denote $1\subset\pi$ the tautological flag on L, then the total complex of $\mathscr{L}(\mathscr{M}_i)$ is a right resolution of the bundle $\Lambda^{i-1}(\pi/l)\otimes l$.

4.16. Consider on $L\times L$ the following double complex of sheaves $C^{\cdot\cdot}$:

$$\begin{tikzcd}
\cdots \arrow[r] & \Psi_{10}\boxtimes\mathcal{O}(-1.0) \arrow[r] & \mathcal{O}_{L\times L} \\
\cdots \arrow[r] & \Psi_{11}\boxtimes\mathcal{O}(-1,-1) \arrow[r] \arrow[u] & \Psi_{01}\boxtimes\mathcal{O}(0,-1) \arrow[u] \\
& \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

where the horizontal differentials equal to $\sum l^*_{\xi_i}\otimes l_{x_i}$ and the vertical ones – to $\sum l^*_{\eta_i}\otimes l_{y_i}$. Here ξ_i is a base of $E=A_{10}$, x_i – the dual base of $E^*=H^0(L, \mathcal{O}(1,0))$, η_i – a base of $E^*=A_{01}$, y_i – the dual base of $E=H^0(L, \mathcal{O}(0,1))$.

4.17. Proposition. *The total complex of $C^{\cdot\cdot}$ is a left resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta\subset L\times L$. If we truncate $C^{\cdot\cdot}$, deleting all terms except those belonging to the intersection of the first $N-1$ rows (from 0-th up to $N-2$-th) and the first $N-2$ columns, then the total complex of the remaining double complex will have, apart from $H^0=\mathcal{O}_\Delta$, only one non-trivial cohomology sheaf, and namely in the last term. This cohomology sheaf (i.e. the kernel) is isomorphic to*

$$\bigoplus_{i=0}^{N-1}\Lambda^i(\pi/l)(-1,0)\boxtimes\Lambda^i(\pi/l)^*(-N+2,-N+1). \quad \square$$

Hence we obtain a finite resolution of the diagonal on $L\times L$.

Consider the following sets of sheaves on L:

$$Y'=\{\mathcal{O}(i,j),\ -N+2\leqq i,j\leqq 0,\ \Lambda^i(\pi/l)^*(-N+2,-N+1),\ 0\leqq i\leqq N-2\},$$
$$X'=\{\Psi_{ij},\ -N+2\leqq i,j\leqq 0,\ \Lambda^i(\pi/l)(-1,0),\ 0\leqq i\leqq N-2\}.$$

Introduce on Y' a partial order, setting $\mathcal{O}(i,j)\leqq\mathcal{O}(k,l)$, iff $i\geqq k$ and $j\geqq l$, and setting the sheaves $\Lambda^i(\pi/l)^*(-N+2,-N+1)$ to be incomparable with each other and less than $\mathcal{O}(-N+2,-N+2)$. Similarly introduce a partial order on X'.

4.18. Proposition. *If $\mathscr{F}, \mathscr{G}$ are elements of X' (resp. Y'), then* $\operatorname{Ext}^p(\mathscr{F},\mathscr{G})=0$ *for* $p>0$, $\operatorname{Hom}(\mathscr{F},\mathscr{G})=0$ *unless* $\mathscr{F}\leqq\mathscr{G}$, *and* $\operatorname{Hom}(\mathscr{F},\mathscr{F})=\mathbb{C}$. $\square$

Denote $\mathfrak{A}'$ and $\mathfrak{B}'$ the full subcategories in $\mathrm{Sh}(L)$ on the sets of objects X' and Y'.

4.19. Theorem. *$D^b(L)$ is equivalent to* $\mathrm{Tr}(\mathfrak{A}')$ *and* $\mathrm{Tr}(\mathfrak{B}')$ *as a triangulated category.* □

4.20. The poset Y' is ranked with rank function l: $l(\mathcal{O}(i,j))=i+j$, $l(\Lambda^i(\pi/l)^*(-N+2, -N+1))=-2N+3$. Denote α the bijection $Y'\to X'$, sending $\mathcal{O}(-i,-j)$ to Ψ_{ij}, $\Lambda^i(\pi/l)^*(-N+2, -N+1)$ – to $\Lambda^i(\pi/l)(-1,0)$. Then the following orthogonality relation holds:

$$H^i(L, \mathcal{F}\otimes\mathcal{G})=\begin{cases}0 & \text{for all } i, \text{ if } \mathcal{G}\neq\alpha(\mathcal{F})\\ 0 & \text{for } i\neq -l(\mathcal{F}), \text{ if } \mathcal{G}=\alpha(\mathcal{F})\\ \mathbb{C} & \text{for } i=-l(\mathcal{F}), \text{ if } \mathcal{G}=\alpha(\mathcal{F}).\end{cases}$$

Hence for $\mathcal{F}\in Y'$ the complex $\hat{\mathcal{F}}$ is quasi-isomorphic to $\alpha(\mathcal{F})[-l(\mathcal{F})]$. Introducing, analogously to n. 4.12, the graded categories $\mathfrak{A}'_g$ and $\mathfrak{B}'_g$, we find that they are Koszul categories, Priddy dual to each other.

4.21. Examples. If $N=3$, then the set Y' contains only invertible sheaves: $Y'=\{\mathcal{O}(i,j)\}$, where $(i,j)\in\{(-1,-2), (-2,-1), (-1,-1), (-1,0), (0,-1), (0,0)\}$.

§ 5. Representations of parabolic subgroups in GL$(N, \mathbb{C})$

5.1. Let $P=P(i_1, \ldots, i_k, n)\subset \mathrm{GL}(N)$ be the parabolic subgroup, the stabilizer of a flag of subspaces $V_{i_1}\subset V_{i_2}\subset\ldots\subset V_{i_k}\subset V_n$, $\dim V_j=j$. Consider the set $\tilde{X}=\tilde{X}(i_1, \ldots, i_k, n)$ of representations of P, consisting of

$$\Sigma^{\alpha_1}v_{i_1}\otimes\Sigma^{\alpha_2}V_{i_2}\otimes\ldots\otimes\Sigma^{\alpha_k}V_{i_k}\otimes\Sigma^{\alpha_{k+1}}V_n,$$

where α_j, $j=1, \ldots, k$, runs over Young diagrams with no more than i_j rows and no more than $i_{j+1}-i_j$ columns (we set $i_{k+1}=n$), and α_{k+1} is an arbitrary non-increasing senuence of n integers. Denote P-mod the category of all finite-dimensional representations of P as an algebraic group and $\tilde{\mathfrak{A}}=\tilde{\mathfrak{A}}(i_1, \ldots, i_k, n)$ – the full subcategory in P-mod on the set of objects $\tilde{X}$. The aim of this section is to prove the following fact.

5.2. Theorem. *$D^b(P$-mod$)$ is equivalent to* $\mathrm{Tr}(\tilde{\mathfrak{A}})$ *as a triangulated category.*

The proof relies on the consideration for each $W\in\mathrm{Ob}\,P$-mod of the corresponding homogenous vector bundle $\tilde{W}$ on the flag variety $F(i_1, \ldots, i_k, n, \mathbb{C}^N)$, where N is sufficiently large. We shall use the following well-known fact.

5.3. Lemma. *An algebraic action of* GL$(N, \mathbb{C})$ *by affine transformations on an affine space has a fixed point.* □

The canonical resolution (3.3) of the bundle $\tilde{W}$ does not, in general, yield a complex of representations of P, and by two reasons. First, the spaces $H^i(F, \tilde{W}\otimes\Phi_a)$ can have a non-trivial structure of GL(N)-module. This non-triviality, however, will be extinct for N sufficiently large (Lemma 5.5 below).

Secondly, the resolution (3.3) is canonical only up to homotopy equivalence, and the possibility to choose a homogenous resolution for a homogenous complex is yet to be proved.

5.4. Lemma. *A homogeneous complex $\mathscr{F}^{\bullet}$ on $F=F(i_1, \ldots, i_k, n, \mathbb{C}^N)$ has, in the homotopy class of its canonical resolution* (3.3), *a homogenous representative.*

Proof. In § 3, we have constructed a true complex over $D^b(F)$

$$\ldots \to \bigoplus_{l(\alpha)=i} R\Gamma(F, \mathscr{F}^{\bullet} \otimes \Phi_\alpha)[i-|\alpha|] \otimes E_\alpha \to \ldots \tag{5.1}$$

and a canonical Postnikov system in $D^b(F)$, attached to this complex, which realises $\mathscr{F}^{\bullet}$ as its convolution. Take for $R\Gamma$ the graded space of hypercohomology with zero differentials and denote Z^i the i-th term of the complex (5.1) thus understood ($i \leqq 0$). Each Z^i is equipped with a natural $\mathrm{GL}(N)$-action, and we have to choose equivariant representatives for B-terms of our Postnikov system. To do this, we proceed as explained in n. 3.3 (on the example of Grassmann varieties), but a bit more carefully. At the i-th step we have a well-defined homotopy class of morphisms $Z^{-i} \to B^{-i}$. This homotopy class is an affine space. If we assume by induction that B^{-i} is already represented by a homogenous complex, then this affine space acquires a $\mathrm{GL}(N)$-action. By Lemma 5.3, this action has a fixed point. We define B^{-i-1} as the cone of the morphism, corresponding to this fixed point. Thus proceeding, we obtain a desired homogenous resolution. □

Now, suppose given a finite complex $W^{\bullet} \in \mathrm{Ob}\, D^b(P\text{-mod})$. Consider the complex $W^{\bullet} \otimes (\det V_n)^{\otimes m}$, where m is large enough (with respect to $W^{\bullet}$) and consider the corresponding homogenous vector bundle $(W^{\bullet} \otimes (\det V_n)^{\otimes m})^{\sim}$ on $F(i_1, \ldots, i_k, n, \mathbb{C}^N)$, where N is large enough with respect to $W^{\bullet}$ and m.

5.5. Lemma. *For every $W^{\bullet} \in \mathrm{Ob}\, D^b(P\text{-mod})$ there exists $m_0 \in \mathbb{N}$ such that $\forall m \geqq m_0\ \exists N_0 \in \mathbb{N}$ such that $\forall N \geqq N_0$ the homogenous resolution of the complex $(W^{\bullet} \otimes (\det V_n)^{\otimes m})^{\sim}$ on $F(i_1, \ldots, i_k, n, \mathbb{C}^N)$ originates from a complex of representations of the group P, which we denote $R^{\bullet}(W^{\bullet}, m, N)$. The complex $R^{\bullet}(W^{\bullet}, m, N) \otimes (\det V_n^*)^{\otimes m}$ does not depends, modulo the homotopy equivalence, from the choice of $m \geqq m_0$, $N \geqq N_0$, nor from the choice of equivariant differentials in the resolution.*

Proof. First consider some $N > n$. The terms of the canonical resolution of the complex $\tilde{W}^{\bullet}$ are sums of summands of the form $\mathbb{H}^i(F(i_1, \ldots, i_k, n, \mathbb{C}^N, \tilde{W}^{\bullet} \otimes \Phi_\alpha) \otimes E_\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_{k+1})$ is a sequence of Young diagrams; α_1 has $\leqq (i_2 - i_1)$ columns, α_2 has $\leqq (i_3 - i_2)$ columns, $\ldots$, α_k has $\leqq (n - i_k)$ columns, α_{k+1} has $\leqq (N-n)$ columns. If we ensure the circumstance that all arising $\mathbb{H}^i$ have trivial $\mathrm{GL}(N)$-module structure, then the homogenous resolution, consisting of such terms, must originate from a complex of representations of P. To ensure this circumstance, consider the hyperdirect images of $\tilde{W}^{\bullet} \otimes \Phi_\alpha$ in the projection on the Grassmanian $G(n, \mathbb{C}^N)$. From the structure of the bundles

Φ_α we see that they are direct sums of some $\Sigma^\gamma S_n \otimes \Sigma^{\alpha_{k+1}^t} S_n^\dagger$, where γ are some dominant (not necessarily positive) weights. The multiplicity, with which such a summand enters into a given hyperdirect image, does not depend on the choice of $N > n$. If we change $W^\cdot$ to $W^\cdot \otimes \det(V_n)$, then each γ changes to $(\gamma_1 + 1, \ldots, \gamma_n + 1)$. Now, let us perform a twist by $(\det(V_n)^{\otimes m}$ so as to make all γ_j positive. After that, choose $N > n$ so large, that all obtained Young diagrams were contained in the rectangle $n \times (N-n)$. Then, from the Lemma 3.2 b), we find that in the cohomology of all the hyperdirect images the group $\mathrm{GL}(N)$ acts trivially. Using Leray spectral sequence, we see that the $\mathrm{GL}(N)$-action in $\mathbb{H}^i(F, (W^\cdot \otimes (\det(V_n)^{\otimes m})^\sim \otimes \Phi_\alpha$ is also trivial, due to semi-simplicity of representations of $\mathrm{GL}(N)$.

Lemma 5.4 gives now a complex of representations of P, consisting of direct sums of elements of the set $\tilde{X}$ and quasi-isomorphic to $W^\cdot \otimes (\det V_n)^{\otimes m}$. Since the transformation $(\det(V_n)) \otimes ?$ preserves the set $\tilde{X}$, the twist can be inverted, and we obtain a resolution of $W^\cdot$. A different choice of differentials in the homogenous resolution would lead to homotopy equivalence at the level of representations of P. The independence (modulo homotopy equivalence) of the obtained complex from the choice of m and N is also clear. □

5.6. Given a morphism $\phi: W_1 \to W_2$ in $D^b(P\text{-mod})$, we can represent the morphism $\tilde{\phi}: \tilde{W}_1 \to \tilde{W}_2$ by an equivariant morphism of homogenous resolutions, using Lemma 5.3. Indeed, let U_i be homogenous resolutions of $\tilde{W}_i$ (N being large enough), given by Lemma 5.4. The morphism in question defines a correct homotopy class of morphisms of complexes $U_1 \to U_2$. This class is an affine space, in which acts $\mathrm{GL}(N)$. A fixed point in this class is the desired equivariant morphism.

It follows that higher Ext's between elements of $\tilde{X}$ in the category P-mod vanish. To see this, consider a diagram $W_1 \xleftarrow{q} J \to W_2[i]$, in $\mathrm{Hot}(P\text{-mod})$, representing an element of $\mathrm{Ext}^i(W_1, W_2)$ (here q is a quasi-isomorphism, $W^i \in \tilde{X}$). Tensor it by $(\det(V_n))^{\otimes m}$ for m large enough and consider the corresponding diagram of equivariant morphisms of homogenous complexes on $F(i_1, \ldots, i_k, n, \mathbb{C}^N)$ for N large enough:

$$(W_1 \otimes (\det V_n)^{\otimes m})^\sim \leftarrow (J \otimes (\det V_n)^{\otimes m})^\sim \to (W_2[i] \otimes (\det V_n)^{\otimes m})^\sim .$$

Choose homogenous resolutions for all these complexes. The resolutions for $(W_i \otimes (\det V_n)^{\otimes m})^\sim$ will be themselves; denote U the homogenous resolution for the middle term. Now, represent the morphisms by equivariant morphisms of the resolutions. If N is chosen sufficiently large, then all the terms of the resolutions considered will belong to the set $X(i_1, \ldots, i_k, n, N)$. Since the left arrow is a quasi-isomorphism, U must coincide with $(W_1 \otimes (\det(V_n)^{\otimes m})^\sim$. Hence, the right arrow must be zero, as well as the element of Ext^i represented by our diagram.

So, we obtain, by Proposition 2.5, an embedding $\mathrm{Tr}(\mathfrak{A})$ into $D^b(P\text{-mod})$ as a full triangulated subcategory. It follows from Lemma 5.5, that every object of $D^b(P\text{-mod})$ is isomorphic to the image of some object of $\mathrm{Tr}(\mathfrak{A})$. Hence, our embedding is in fact an equivalence and Theorem 5.2 is proved.

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