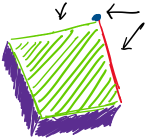
GKM Theory for Flag Varieties, Chevalley-Monk Formula

Wednesday, December 2, 2020 5:07 PM

- Introduction to Flag Varieties & Main Problem (in type A)
- GKM Theory, Permutation Actions on $H_T^*(\mathcal{F}l_n)$
- Chevalley-Monk Formula



(Complete) Flag Varieties

(or $\mathbb{R}^n$)

Def. A complete flag $F_\bullet = \{0\} = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_n = \mathbb{C}^n$ is a nested sequence of vector spaces F_i, $0 \le i \le n$ where $\dim(F_i) = i$.

We can represent a flag variety $F_\bullet$ by an ordered basis of $\mathbb{C}^n$ – so that if $\langle f_1, \dots, f_n \rangle = \mathbb{C}^n$ (wrt some fixed basis $\{e_i\}$), then $F_i = \langle f_1, \dots, f_i \rangle$, i.e. F_i is spanned by the first i rows in the matrix

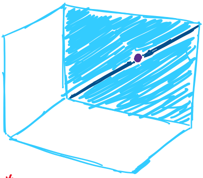
$$\begin{bmatrix} - f_1 - \\ - f_2 - \\ \vdots \\ - f_n - \end{bmatrix}$$

← F_i is the span of the first i rows.

Ex: $F_\bullet = \langle 6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4, e_2 \rangle$

We can represent $F_\bullet$ with a matrix $F_\bullet \in GL_n(\mathbb{C})$*:

$$\begin{array}{cc} \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} & \\ \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & \boxed{1} & 1 \\ 0 & \boxed{1} & 0 & 0 \end{bmatrix} & \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} \end{array}$$



*GL_n because the matrix must have full rank!

$$\begin{bmatrix} 2 & \boxed{1} & 0 & 0 \\ 2 & 0 & \boxed{1} & 0 \\ 7 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

We would like to "simplify" this somehow. Or perhaps, find a "nice" cellular decomposition of $\mathcal{F}l_n(\mathbb{C})$ analogous with Schubert cells for the Grassmannians.

$$= \underbrace{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}}\underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

$\longrightarrow$ Lower-Δ matrix   matrix in "canonical form"

$$\approx \langle 2e_1+e_2,\ 2e_1+e_3,\ 7e_1+e_4,\ e_1 \rangle$$

So any flag can have its basis vectors "rotated" a certain way using a lower-Δ matrix to obtain a matrix in canonical form.

Also, noticing that multiplying on the *left* by a lower triangular matrix doesn't change the flag, we can say

$$\mathcal{F}l_n(\mathbb{C}) \cong GL_n(\mathbb{C})/B$$

(B is called a Borel subgroup)

where B is the subgroup $B \leq GL_n(\mathbb{C})$ consisting of lower-Δ

(equiv: upper-Δ as well)

Reducing in this way always yields 0's to the right and below of the "leading 1's".

$$\begin{bmatrix} 2 & \boxed{1} & 0 & 0 \\ 7 & 0 & \boxed{1} & 0 \\ 7 & 0 & 0 & \boxed{1} \\ \boxed{1} & 0 & 0 & 0 \end{bmatrix}$$

... kind of looks like a permutation matrix with some extra numbers

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline \rightarrow & * & \boxed{1} & 0 & 0 \\ \rightarrow & * & 0 & \boxed{1} & 0 \\ \rightarrow & * & 0 & 0 & \boxed{1} \\ \rightarrow & \boxed{1} & 0 & 0 & 0 \end{array}$$

← This permutation, 2341, is the position of $F_\bullet$ wrt reference basis $\{e_i\}$

$* \in \mathbb{C}$

Schubert Cells $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$.

($E_\bullet$ is called the reference flag)

The Schubert Cell $C_w(E_\bullet)$ is defined by

$(w \in S_n)$

$$C_w(E_\bullet) = \text{All flags with } \underbrace{\text{position}(E_\bullet, F_\bullet)}\ = w$$

$$= \left\{ F_\bullet \in \mathcal{F}l_n \ \middle|\ \dim(E_i \cap F_j) = \operatorname{rank}(w[i;j]) \right\}$$

This definition is confusing, so use this one:

Ex: C_{3421}

$$\nearrow\ \cong$$

$$\mathbb{C}^5$$

$$\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \\ * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $* \in \mathbb{C}$

3421

Note: $C_w \cong \mathbb{C}^{\#\text{ of stars}} \cong \mathbb{C}^{\operatorname{inv}(w)}$

$$\begin{matrix} * & \text{———} & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{matrix}$$

$C_w = w\cdot B$, the B-orbit of the permutation of w:

$$\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\begin{bmatrix}b_{11}&0&0&0\\b_{21}&b_{22}&0&0\\b_{31}&b_{32}&b_{33}&0\\b_{41}&b_{42}&b_{43}&b_{44}\end{bmatrix}=\begin{bmatrix}b_{21}&\boxed{b_{22}}&0&0\\b_{31}&b_{32}&\boxed{b_{73}}&0\\b_{41}&b_{42}&b_{43}&b_{44}\\b_{11}&0&0&0\end{bmatrix}\to\begin{bmatrix}b_{21}/b_{22}&1&0&0\\b_{31}/b_{33}&0&1&0\\b_{41}/b_{44}&0&0&1\\1&0&0&0\end{bmatrix}$$

SCHUBERT VARIETIES

$$X_w(E_\bullet)=\overline{C_w(E_\bullet)}$$

(closure under Zariski topology)

These get kind of weird, however — for example,

$$\begin{bmatrix}\textcircled{1}&0&0&0\\0&*&\textcircled{1}&0\\0&*&0&\textcircled{1}\\0&\textcircled{1}&0&0\end{bmatrix}\in X_{2341}(E_\bullet)=\overline{\left\{\begin{bmatrix}*&\textcircled{1}&0&0\\ *&0&\textcircled{1}&0\\ *&0&0&\textcircled{1}\\ \textcircled{1}&0&0&0\end{bmatrix}\right\}}$$

1342

Now, why might that be?

$$\begin{bmatrix}a&1&0&0\\b&0&1&0\\c&0&0&1\\1&0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}\textcircled{1}&1/a&0&0\\b&0&1&0\\c&0&0&1\\1&0&0&0\end{bmatrix}\to\begin{bmatrix}1&1/a&0&0\\0&-b/a&1&0\\0&-c/a&0&1\\0&-1/a&0&0\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1&1/a&0&0\\0&-b/a&1&0\\0&-c/a&0&1\\0&1&0&0\end{bmatrix}$$

So I can send $a\to\infty$, and picking $\boxed{b=-\alpha a}$ and $\boxed{c=-\beta a}$, I can obtain

$$\xrightarrow[(b\to\infty)\ (c\to\infty)]{\text{"}a\to\infty\text{"}}\begin{bmatrix}1&0&0&0\\0&\alpha&1&0\\0&\beta&0&1\\0&1&0&0\end{bmatrix}.$$

$$\overline{X_w}=\bigcup_{\underset{\text{Bruhat order}}{\underset{\uparrow}{v\leq w}}}C_w$$

GKM THEORY APPLIED TO FLAG VARIETIES.

BIG open problem — given two Schubert classes $[X_\pi]\cdot[X_\sigma]\in H^*(\mathcal{F}l_n(\mathbb{C}))$, what are the intersection numbers X_w in the expansion below?

$$[X_\pi]\underset{\substack{\uparrow \\ \text{cup product}}}{}[X_\sigma] = \sum_\omega{}' c_{\pi,\sigma}^{\omega}[X_\omega]$$

One approach is to use GKM Theory.

TORUS ACTION ON $\mathcal{F}\ell_n(\mathbb{C})$

In type A, the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ is given by

$GL_n(\mathbb{C})/B \leftarrow$ the matrix subgroup of lower (or upper)-Δ matrices

Identify the torus $T=(\mathbb{C}^*)^n$ with the n×n diagonal matrices:

$$t=\underline{(t_1,\dots,t_n)} \longleftrightarrow \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix} = t$$

$t_i \in \mathbb{C}^*$

We can then define the torus action as follows:

$$\longrightarrow t\cdot(F_0 \subseteq F_1 \subseteq \dots \subseteq F_n) = tF_0 \subseteq tF_1 \subseteq \dots \subseteq tF_n$$

OR identify the flag with the appropriate coset representative $[g]$, then

$$t\cdot[g] = [tg] \qquad [g] \in GL_n(\mathbb{C})/B$$

LEMMA. The T-fixed flags are precisely the permutation flags $[\omega]$! $\in S_n$

PROOF. Since T must fix F_1, the first row must look like:

$$[0 \dots 0\ 1\ 0 \dots 0] \qquad \begin{bmatrix} 0 \dots 0 & 1 & \dots & a & \dots \\ & t_1 & & at_2 & \end{bmatrix} \qquad \begin{bmatrix} & & 1 & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \end{bmatrix}$$

Since T must fix F_2, then the second row must look like:

$$\begin{bmatrix} & & 1 & a^{t_2/t_1} \end{bmatrix}$$

$$\begin{bmatrix} 0 \dots 0 & \boxed{1} & 0 \dots & 0 \\ 0 \dots 1\ 0 & \downarrow & \dots & 0 \end{bmatrix}$$

...and so on.

(1) $\underset{\sim}{\underline{\mathcal{F}\ell_n(\mathbb{C})}}$ has finitely many T-fixed points.

LEMMA. The one-dimensional Torus orbits look like permutation flags with one additional nonzero entry. (kind of)

Example:

$$\begin{bmatrix} a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(t_1,t_2,t_3,t_4)} \begin{bmatrix} t_1a & t_2 & 0 & 0 \\ t_1b & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \\ t_1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \frac{t_4}{t_2}a & 1 & 0 & 0 \\ \frac{t_4}{t_3}b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} * & & \\ * & & \\ * & & \end{bmatrix}$

Two dimensional orbit //

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{a\to 0} \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{a\in\mathbb{C}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

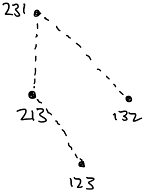
$\omega = 213$

$$\begin{bmatrix} 1 & 1/a & 0 \\ 0 & -1/a & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1/a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xleftarrow{b\to 0} \begin{bmatrix} b & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\omega = 231$ 132

$$\begin{bmatrix} 1 & 1/b & 0 \\ 0 & 0 & 1 \\ 0 & -1/b & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 1 & 0 \\ c & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\omega = 231$

Note: The closures of each of these orbits is homeomorphic to $\mathbb{CP}^1$.

When are two T-fixed flags connected by a 1-D Torus orbit $\mathcal{O}$?

When they differ by a single two-cycle.

$\Updownarrow$

$[\omega] = [(ij)\omega']$

PROOF.

$$\begin{matrix} i & * & 1 \\ j & 1 & \end{matrix} \to \begin{matrix} a & 1 \\ 1 & \end{matrix} \to \begin{matrix} 1 & 1/a \\ 0 & -1/a \end{matrix} \to \begin{matrix} 1 & 1/a \\ 0 & 1 \end{matrix} \xrightarrow{\lim} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

I've swapped rows i and j.

There are $\binom{n}{2}$ transpositions in S_n.

$\Longrightarrow$ (2) $\mathcal{F}\ell_n$ has finitely many 1D T-orbits.

(3) (GORESKY, KOTTWICZ, MACPHERSON).

$\mathcal{F}\ell_n$ is equivariantly formal.

$\Longrightarrow$ $\mathcal{F}\ell_n$ IS A GKM SPACE.

$$\Longrightarrow H_T^*(\mathcal{F}\ell_n) = \left\{ (p_v) \in H_T^*(pt)^{|S_n|} \;\middle|\; \begin{array}{l}\text{TORUS WEIGHT}\\ \text{CONDITION}\end{array} \right\}$$

//

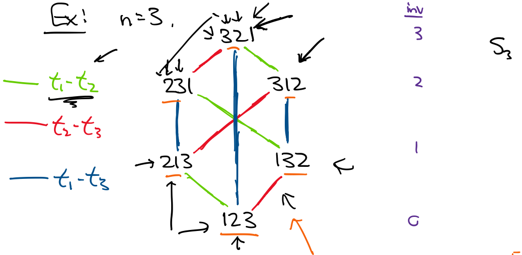
* Caution - left multiplication

$$\left\{ (p_w) \in \mathbb{C}[t_1,\dots,t_n]^{\overset{n!}{\overset{\|}{|S_n|}}} \;\middle|\; \begin{array}{l}\text{If } \overset{\downarrow}{w} = (ij)w', \text{ then}\\ t_i - t_j \mid p_w - p_{w'}\end{array} \right\}$$

MOMENT GRAPH OF $H_T^*(\mathcal{F}\ell_n(\mathbb{C}))$

- Vertices: $w \in S_n$
- Edges: $(i\,j) \in \binom{n}{2}$ ← Transpositions of S_n with label $t_i - t_j$ $(i<j)$
- Direction of flow has $w \longmapsto w'$ if $\mathrm{inv}(w) > \mathrm{inv}(w')$.

Ex: $n=3$.



$n=4$ has 24 vertices...

...$n=5$ has 120...

$$H_T^*(\mathcal{F}\ell_n) = \mathbb{C}[t_1,\dots,t_n]^{|S_n|}$$

Up to some relations

CANONICAL CLASSES.

$\left(\underset{123}{\_\_\_}, \underset{213}{\_\_}, \underset{132}{\_\_}, \_\_, \ , \ \right)$

Recall! If our space is "nice enough" ($\mathcal{F}\ell_n(\mathbb{C})$ is), then the following construction yields a ($H_T^*(pt)$-module) basis of $H_T^*(X)$:

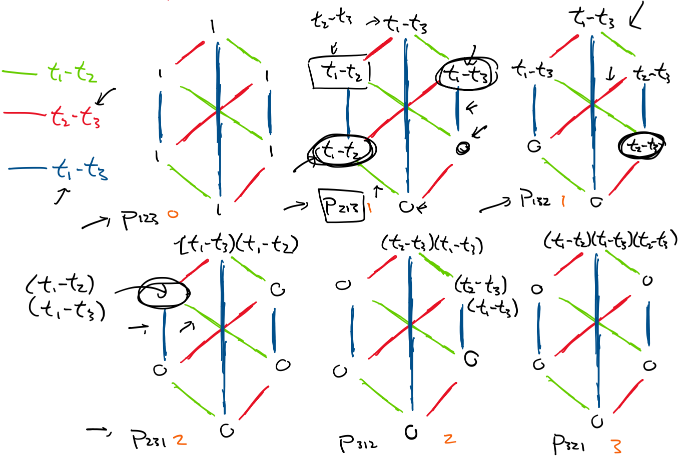
(1) One class p_w for each $w \in X^T$

(2) $p_w|_v = 0$ if there is no directed chain

$$v \mapsto \dots \mapsto w$$

(3) $p_w|_w = \prod_{w \mapsto v}$ (label on edge $w \mapsto v$)

(4) Pick homogeneous polynomials of degree deg $p_w|_w$ that satisfy GKM conditions.



THEOREM. The canonical classes p_v are precisely the Schubert classes.

FACT: The Poincaré polynomial $\mathcal{H}(H_T^*(\mathcal{F}\ell_n))$ is given by

$$\mathcal{H}(H^*(\mathcal{F}\ell_n)) = \sum_{\sigma \in S_n} q^{inv(\sigma)} = (1)(1+q)\dots(1+q+\dots+q^{n-1})$$

$$= [n]_q!$$

DOT & STAR ACTIONS, CHEVALLEY-MONK FORMULA.

Brief review of S_n.

$$S_n = \left\langle s_1,\dots,s_{n-1} \;\middle|\; \begin{array}{l} s_i^2 = 1 \\ s_is_j = s_js_i \text{ if } |i-j|\ge 2 \\ s_is_js_i = s_js_is_j \text{ if } i=j\pm 1 \end{array} \right\rangle$$

- Any permutation $w \in S_n$ can be written as a word of minimal length $w = \prod_i s_{\sigma(i)}$
- $\operatorname{inv}(w) = \min_{\text{word}=w} \ell(\text{word})$

(Tymoczko)

The dot action is defined by: $(w\cdot p)|_v = wp|_{w^{-1}v}$

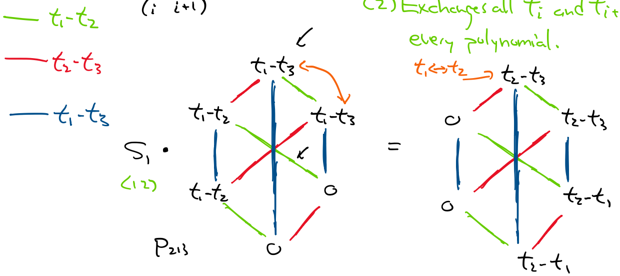
(if $f \in \mathbb{C}[t_1,\dots,t_n]$, $wf = f(t_{w(1)},\dots,t_{w(n)})$)

This definition is perhaps confusing, so look at s_i:

s_i acts on a class by:
$(i \;\; i+1)$

(1) Exchanges all polynomials on either side of an edge $t_i - t_{i+1}$

(2) Exchanges all t_i and t_{i+1} in every polynomial.



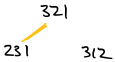
EXERCISE: Compute $s_2 \cdot p_{213}$.

$*$ - ACTION (KOSTANT-KUMAR, S.)

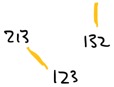
The star action is defined by:

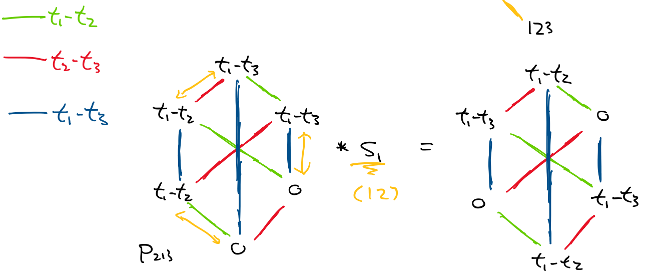
$$(p * w)|_v = p|_{vw}$$

CAREFUL - Star acts on the right.



How does s_i act on p under $*$?





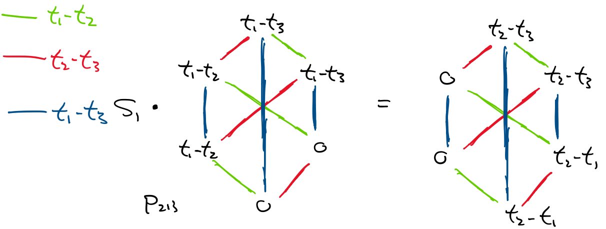
FORMULA FOR DOT.

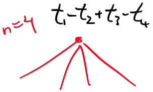
$$s_i \cdot p_w = \begin{cases} p_w - (t_i - t_{i+1})\, p_{s_i w} & \text{if } s_i w < w \text{ in the moment graph} \\ p_w & \text{otherwise} \end{cases}$$

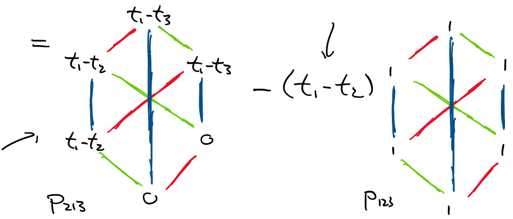
$(i\ i{+}1)$

Ex:

$123 = (12) \cdot 213$







...the star formula is really messy.

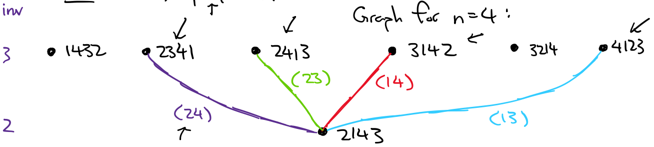
CHEVALLEY–MONK FORMULA. (Pieri Rule for Flags)

$$p_i p_w = \sum_{\substack{(jk):\\ \ell(w(jk)) = \ell(w)+1 \\ j \le i < i+1 \le k}} p_{w(jk)}$$ * Right multiplication!

$\uparrow$ Schubert class of s_i $1 \le i \le n-1$

$P_2 \leftrightarrow (23)$

Ex: $n=4$, $\overset{\uparrow}{P_2} P_{2143}$. Here is a section of the GKM moment Graph for $n=4$:



So that $p_2 p_{2143} = \underbrace{p_{2341} + p_{2413} + p_{3142} + p_{4123}}$

Sketch of proof:

First, notice that in the formula for dot

$$\left[\; s_i \cdot p_w = \begin{cases} \underline{p_w} - \underbrace{(t_i - t_{i+1})} p_{s_i w} & \text{if } s_i w < w \text{ in the moment graph} \\ \underline{p_w} & \text{otherwise} \end{cases} \;\right]$$

that under the forgetful map

$$\begin{tikzcd} H_T^*(\mathcal{F}\ell_n) \arrow[r] \arrow[d] & H^*(\mathcal{F}\ell_n) \arrow[dl, equal] \\ \left(H_T^*(\mathcal{F}\ell_n) \big/ \langle t_1, \dots, t_n \rangle H_T^*(\mathcal{F}\ell_n) \right) & \end{tikzcd}$$

That $\underline{s_i \cdot p_w \cong p_w}$.

$\downarrow$

So then $(w \cdot p_i) p_w \cong p_i p_w$.

So just expand $\uparrow$ in $H_T^*(\mathcal{F}\ell_n)$, and consider only the t^0 degree. □

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