Meeting 11/11

Wednesday, November 11, 2020 12:23 PM

Plan ① Recap on homology

② Recap on cohomology + Poincaré duality

③ A little about Chow

④ Remaining HW problems.

① Recap on homology

Cellular homology

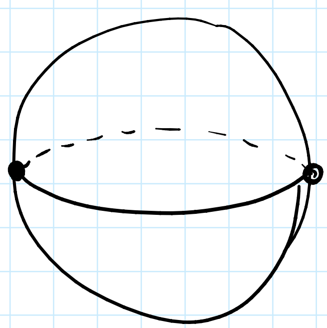
X = cell complex = union of cells

X_0 = bunch of points

$X_1 = X_0$ + some intervals (= 1-disks) glued to X_0 on the boundary

$X_2 = X_1$ + some 2-disks glued to X_1 on the boundary ...

Ex

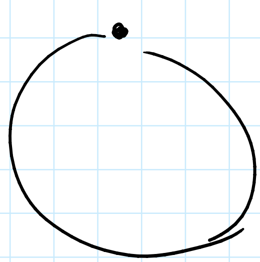


$X_0 = 2$ points

$X_1 = 2$ points + 2 intervals

$X_2 = S^2$

Ex



X_0 = point

$X_1 = X_0$

$X_2 = S^2$ (add 2-cell, no 1-cells).

Homology of X (with fixed cell

two 1-cells).

Homology of X (with fixed cell decomposition)

$$\to C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} C_{i-2} \to \cdots$$

C_i = vector space spanned by i-cells (over $\mathbb{Q}$)

$$\partial(\text{cell}) = \sum_k (\text{cell}_k)\cdot(\text{coef})$$

i-dim

boundary of cell = union of smaller cells.

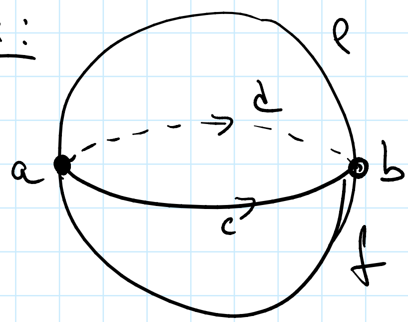
i-1-dim.

Key fact: With appropriate defn. of coefficients

$$\boxed{\partial^2 = 0}$$

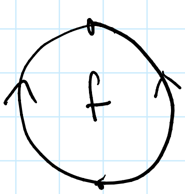
$$H_i(X) = \frac{\operatorname{Ker}(\partial)|_{C_i}}{\operatorname{Im}(\partial)|_{C_{i+1}}} \qquad \underbrace{C_{i+1} \xrightarrow{\partial}}_{\operatorname{Im}} \underbrace{C_i \xrightarrow{\partial} C_{i-1}}_{\operatorname{Ker}}$$

Ex:



$$\begin{array}{ccccc} C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 \\ \| & & \| & & \| \\ \langle e,f\rangle & & \langle c,d\rangle & & \langle a,b\rangle \end{array}$$

2-cells 1-cells 0-cells



c^1 $\partial(c) = b - a$

c^1 $\partial(d) = b - a$

c^2 $\partial(f) = c - d$

c^2 $\partial(e) = c - d$

Check: $\partial^2(e) = \partial(c-d)$

$= (b-a) - (b-a) = 0.$

$H_2 = \operatorname{Ker}(\partial) = \langle e - f\rangle$

$H_2 = \langle e-f\rangle \cong \mathbb{Q}$

$\operatorname{Im}(\partial) = 0$

$H_2 : \mathrm{Ker}(\partial) = \langle e - f \rangle$

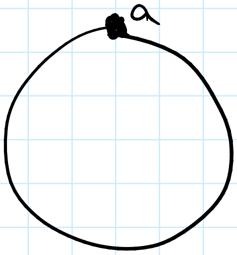
$\mathrm{Im}(\partial) = 0$

$H_2 = \langle e - f \rangle \cong \mathbb{Q}$

$H_1 : \mathrm{Ker}(\partial) = \langle c - d \rangle = \mathrm{Im}(\partial) \Rightarrow H_1 = 0$

$H_0 : \mathrm{Ker}(\partial) = \langle a, b \rangle \quad \mathrm{Im}(\partial) = \langle b - a \rangle \qquad H_0 = \dfrac{\langle a, b\rangle}{\langle b - a \rangle} = \mathbb{Q}\langle a \rangle$

Ex



$$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$
$$\| \qquad\quad \| \qquad\quad \|$$
$$\mathbb{Q} \qquad\quad 0 \qquad\quad \mathbb{Q}$$

$\partial = 0$ (no 1-cells!)

Ker = everything $\quad$ Im = 0

Facts ① H_i do not depend on a choice of cell decomposition.

② If X is smooth, oriented, connected
$\dim = n$

$$H_0(X) = \mathbb{Q} = H_n(X)$$

$H_0(X) = \langle pt \rangle$

$H_n(X)$ = spanned by sum of all n-cells with signs. ($e-f$ in example).

③ If X admits a cell decomposition where all cells have even dimension

then $\begin{cases} H_{2i}(X) = \mathrm{span}(2i\text{-cells}) \\ H_{2i+1}(X) = 0 \end{cases}$ $\qquad$ Proof: $\partial = 0$

Ex: Grassmannians, flag varieties (over $\mathbb{C}$)
$\mathbb{C}P^n$, ...

Proof: Cells = (ex. Schubert cells) are $\mathbb{C}$-vector spaces $\Rightarrow$ even dim.

Warning: For $\mathbb{R}P^n$ or real $Gr(k,n;\mathbb{R})$ tricky boundary maps.

(4) $f: X \longrightarrow Y$ continuous map

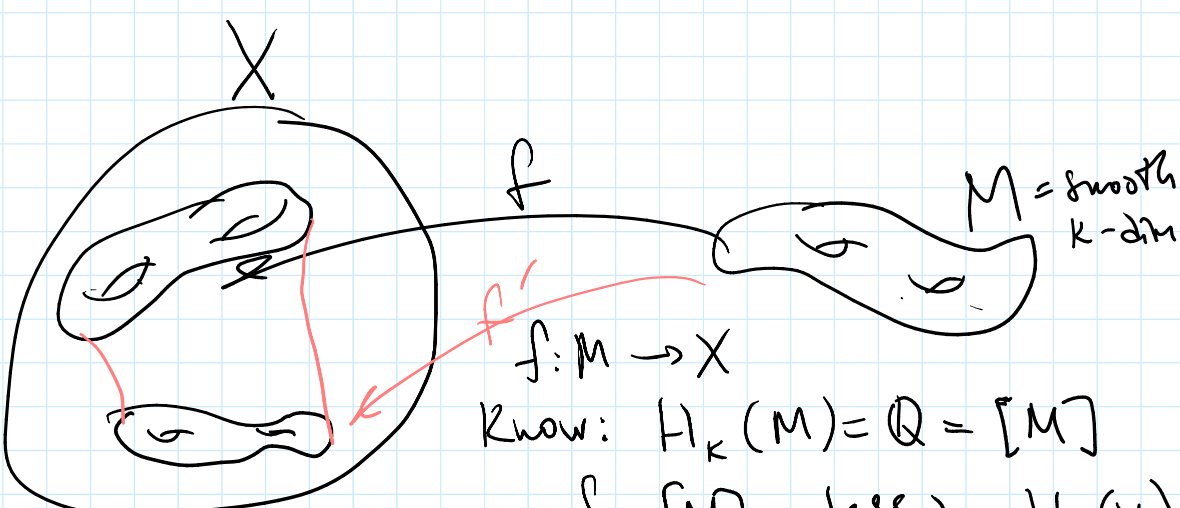
$\Rightarrow f_*: H_i(X) \longrightarrow H_i(Y)$ linear map on homology.

If we perturb f continuously, f_* does not change.

$F: X \times [0,1] \longrightarrow Y$ continuous (homotopy)

$f_t(x) = f(x,t)$, $t \in [0,1]$ $\quad (f_t)_*$ does not depend on t.

(5) We can combine these as follows:



$f: M \to X$

Know: $H_k(M) = \mathbb{Q} = [M]$

$f_* [M] = $ class in $H_*(X)$.

$f_*[M] = $ class in $H_k(X)$.

$f_*: H_k(M) \to H_k(X)$

If f, f' are homotopic, get the same class in $H_k(X)$.

Ex $M \subset X$ smooth subvariety

$f: M \to X$ inclusion $\quad f_*[M] = $ fund. class of M in $H_k(X)$

⑥ Künneth formula

$$H_*(X \times Y) = H_*(X) \otimes H_*(Y)$$

$$H_k(X \times Y) = \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y)$$

II: Cohomology & Poincare duality

$H^i(X) = $ dual vector space to $H_i(X)$

cohomology.

$f: X \to Y$

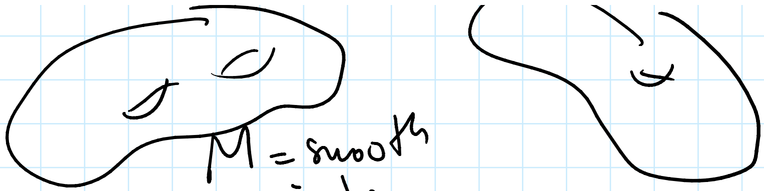
$f_*: H_k(X) \to H_k(Y) \quad \boxed{f^*: H^k(Y) \to H^k(X)}$

note: change direction.

Poincare duality: $X = $ smooth, orientable, $\dim = n$

There is a nondegenerate pairing (intersection pairing)

$$H_i(X) \times H_{n-i}(X) \longrightarrow \mathbb{Q}$$



M = smooth i-dim

N smooth $(n-i)$-dim

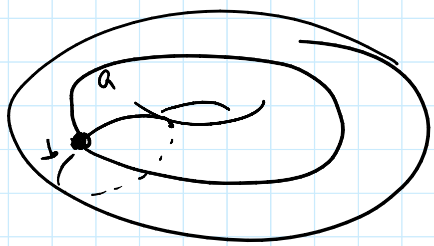
$[M] \times \underbrace{[N]}_{n-i} \longrightarrow$ number of intersection pts (with signs) of M and N.

For fixed N, this defines a linear functional on H_i

Cor $H_{n-i}(X)$ = dual vector space to $H_i(X) \simeq H^i(X)$

Poincaré duality $\boxed{H_{n-i}(X) = H^i(X)}$

Ex



$H_0(T) = \mathbb{Q}$ (pt).

$H_1(T) = \mathbb{Q} \oplus \mathbb{Q}$ (1-cells)

$H_2(T) = \mathbb{Q}$ (2-cell)

$a \cdot a = 0$
$b \cdot b = 0$ $\quad a \cdot b = \pm 1$

Why everyone likes cohomology?

Fact. $H^*(X)$ is an algebra!

There's multiplication $H^i(X) \times H^j(X) \to H^{i+j}(X)$
nice, bilinear

- $f^* : H^i(Y) \to H^i(X)$ is algebra homomorphism (agrees w. multiplication)

- $\alpha \in H^i(X) \longleftrightarrow PD(\alpha) \in H_{n-i}(X)$

• $\alpha \in H^i(X) \longleftrightarrow PD(\alpha) \in H_{n-i}(X)$

$\beta \in H^j(X) \longleftrightarrow PD(\beta) \in H_{n-j}(X)$

cycles.

make them transversal

$\alpha \cdot \beta \in H^{i+j}(X) \longleftrightarrow PD(\alpha) \cap PD(\beta)$

$$PD(\alpha)\cap PD(\beta) = PD(\alpha\cdot\beta)$$

Ex $H^*(S^2) = \langle \underset{\substack{\| \\ PD[S^2] \\ [S^2]\in H_2}}{\overset{H^0 \ni}{1}} , \underset{\substack{\| \\ PD[pt] \\ [pt]\in H_0}}{\overset{H^2 \ni}{a}} \rangle$

$a \cdot a = pt \cap pt = \emptyset$ $\quad a^2 = 0$

Ex: $H^*(\mathbb{CP}^2) = \langle 1, a, b \rangle$ $\quad b = a^2$

• $\mathbb{CP}^2$ has 0, 2, 4 - cells

$a \cdot a = PD[\text{line} \cap \text{line}] = PD[pt] = b$

$H_0 = H_2 = H_4 = \mathbb{Q}$

$a = PD[\mathbb{CP}^1]$

$H^0 = H^2 = H^4 = \mathbb{Q}$, $\quad 1 = H^0$ generator, a, b

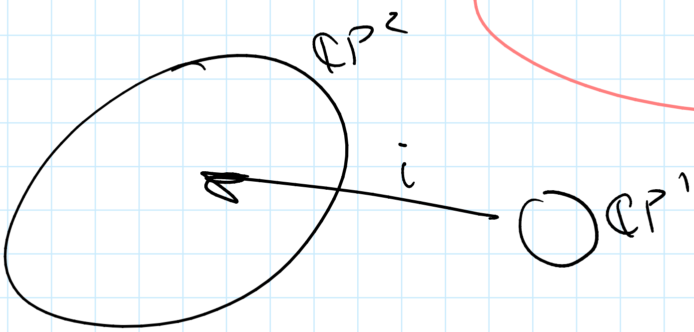
$\mathbb{CP}^1$ = line in $\mathbb{CP}^2$ (union of 0-, and 2-cells)

$\mathbb{CP}^2 = pt \sqcup \underbrace{\boxed{\mathbb{C}}}_{\text{2 cell}} \sqcup \underbrace{\boxed{\mathbb{C}^2}}_{\text{4 cell}}$

$H_2 = \langle \text{class of 2-cell} \rangle$

$\mathbb{CP}^1 = pt \sqcup \mathbb{C} \hookrightarrow \mathbb{CP}^2$

$[\mathbb{CP}^1]$ = class of 2-cell in $H_2(\mathbb{CP}^1)$



$i_*[\mathbb{CP}^1]$ = generator of $H_2(\mathbb{CP}^2)$

generator of $H_2(\mathbb{C}P^2)$

$$\begin{tikzcd} H_0 \; [pt] \arrow[drr, leftrightarrow] & H_2 \; [\mathbb{C}P^1] \arrow[d, leftrightarrow] & H_4 \; [\mathbb{C}P^2] \arrow[dll, leftrightarrow] \\ H^0 & H^2 & H^4 \end{tikzcd} \quad PD$$

$1 = PD[\mathbb{C}P^2]$ $\quad a = PD[\mathbb{C}P^1]$ $\quad b = PD[pt]$

a = generator of H^2 $\quad b$ = generator of H^4

Ex $\mathbb{C}P^n = pt \sqcup \mathbb{C} \sqcup \mathbb{C}^2 \sqcup \dots \sqcup \mathbb{C}^n$

$$H_0 = H_2 = H_4 = \dots = H_{2n} = \mathbb{Q}$$

odd $= 0$ $\qquad \updownarrow PD$

$$H^0 = H^2 = H^4 = \dots = H^{2n} = \mathbb{Q}$$

$H^0 = 1$, $H^2 = \langle a \rangle$

$$a = PD[\underset{\mathbb{C}P^{n-1} \,\in\, H_{2n-2}}{\text{hyperplane}}]$$

$$a^2 = PD[\text{hyperplane} \cap \text{hyperplane}]$$

$\text{hyperplane} \cap \text{hyperplane} = \mathbb{C}P^{n-2}$ (codim 2 subspace)

$$a^k = PD[\mathbb{C}P^{n-k}]$$

Ex $\{f = 0\} \cap \{g = 0\} \cap \{h = 0\} \subset$

$A = \{f=0\}$, $B = \{g=0\}$: 3, 7, 11 hypersurfaces in $\mathbb{C}P^3$

Ex: hypersurfaces in $\mathbb{CP}^3$

$\deg(f)=3 \quad \deg(g)=7 \quad \deg(h)=11.$

How many intersection points are there?

$[B],[C],[A] \in H_4(\mathbb{CP}^3) = \mathbb{Q}$

$PD[A] \in H^2(\mathbb{CP}^3) = m\cdot a$

$PD[A]\cdot a^2 = PD[A]\cdot PD[\text{line}] = A \cap \text{line} = 3\text{ pts.}$

$m\cdot a \cdot a^2 = 3a^3 \Rightarrow m=3$

$PD[A]=3a \quad PD[B]=7a \quad PD[C]=11a$

$PD[A\cap B\cap C] = 3a\cdot 7a\cdot 11a = 3\cdot 7\cdot 11\, a^3$

$A\cap B\cap C = 3\cdot 7\cdot 11$ points

III Chow ring

Idea: model all of the above in the language of alg. geometry

<u>Q</u>: Which classes in $H_*(X)$ are realized by <u>algebraic</u> cycles (fund. classes of algebraic subvar.)

<u>Ex</u> $\{y^2z = x(x-1)(x+1)\} = X$

Ex $\{y^2 z = x(x-1)(x+1)\} = X$
homogeneous cubic equation in x,y,z
$\longrightarrow$ cubic curve in $\mathbb{CP}^2$

Exercise Topologically, this is a 2-torus.
Note: complex 1-dimensional
$\Rightarrow$ real 2 dimensional $\Rightarrow$ surface.

Exercise: compute χ and check it is torus.

alg. cycles in X = whatever is defined in X by alg. equations.

$[X] \in H_2$ ok
$[pt] \in H_0$ $\leftarrow$ add one more equation, intersects X in some number of points.

Ex $X \cap \text{line} = \{3 \text{ pts}\}$

But we can never get any cycle in $H_1(X)$!!

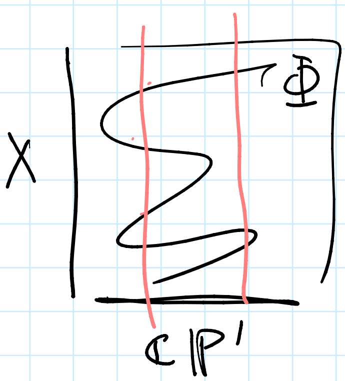
In general: • Cannot realize odd homology by algebraic cycles.
• Not every even-dim homology class can be realized (ex. not every class $H_2(X \times X)$ can be realized)

[Hodge theory]

[Hodge theory] realized)

- Hodge 'conjecture': precise conj. on which classes can be realized

$$A_i(X) = \frac{\left\{ \begin{array}{l} \text{all fund. cycles} \\ \text{of alg. subvarieties} \\ \text{of } \dim_{\mathbb{C}} i \end{array} \right\}}{(\text{rational equivalence})} \longrightarrow H_{2i}(X)$$



$W_0 \sim W_1$ if there's a algebraic subvariety Φ in $X \times \mathbb{P}^1$ such that

$\Phi \cap X \times \{0\} = W_0$

$\Phi \cap X \times \{\infty\} = W_1$

Note: $A_0(\text{torus}) =$ big and uncountable

Not every two points on a torus are equivalent in Chow!!

Fact If X is paved by affine spaces (ex: $\mathbb{C}P^n$, $Gr(k,n)$ ...)

Then $A_*(X) \longrightarrow H_*(X)$ is an isomorphism!!

$$\mathbb{C}P^\infty = \text{"}\lim_{n\to\infty} \mathbb{C}P^n\text{"} = pt \sqcup \mathbb{C} \sqcup \mathbb{C}^2 \sqcup \dots$$

$$H_0 = H_2 = H_4 = \dots = \mathbb{Q}$$

$$\begin{array}{ccccccccc} H^0 & = & H^2 & = & H^4 & = & \dots & = & \mathbb{Q} \\ \| & & \| & & \| & & & & \\ 1 & & a & & a^2 & & \dots & & \end{array}$$

$$H^*(\mathbb{C}P^\infty) = \mathbb{C}[a] \quad a \in H^2.$$

$$H^*(\mathbb{C}P^n) \simeq \mathbb{C}[a]/(a^n)$$