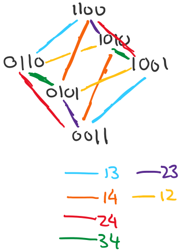
TORUS EQUIVARIANT COHOMOLOGY 8

Thursday, November 5, 2020 10:48 PM

GKM-THEORY.

- BRIEF OVERVIEW OF THE FUNCTOR $H_T^*(-)$
- GKM-THEORY FOR GRASSMANNIANS
  - Tao-Knutson puzzles
- GKM THEORY FOR FLAG VARIETIES

MOMENT GRAPH OF $Gr(2,4)$



TORUS-EQUIVARIANT COHOMOLOGY. (Why?)

For the purposes of this talk, we could treat $H_T^*(-)$ as a contravariant functor $H_T^*: \underline{T\text{-Top}} \longrightarrow \underline{Gr\text{-Ring}}$ *

Topological spaces w/ cont. T-action   Graded Rings

such that: (X is a compact oriented manifold)

(1) $H_T^*(pt) \cong \mathbb{C}[t_1,\dots,t_n]$

(2) There is a nice "forgetful" map $H_T^*(X) \longrightarrow H^*(X)$ which factors as

$$H_T^*(X) \twoheadrightarrow H_T^*(X) \otimes_{H_T^*(pt)} \mathbb{C} \hookrightarrow H^*(X)$$

(here we "tensor out" the Torus Variables)

We would like $\downarrow$, but we need a few more things.

$$H^*(X) \cong H_T^*(X) \big/ \langle t_1,\dots,t_n \rangle H_T^*(X)$$

(3) We have maps

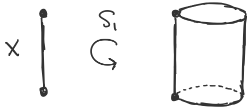
$$\begin{tikzcd} H_T^*(pt) \arrow[r] & H_T^*(X) \arrow[r] & H_T^*(X^T) \arrow[d, phantom, "\cong" rotate=90] \\ & & H^*(X^T)\otimes_{\mathbb{C}} H_T^*(pt) \end{tikzcd}$$

induced by $X^T \hookrightarrow X \twoheadrightarrow pt$.

(4) Among other nice properties.

BRIEF INTRO TO EQUIV. COHOMOLOGY.

Idea: Take X for a spin with a torus T $\longrightarrow$ gives us more "room" to work with, and maybe we recover some information about X.



Definition of T-Equivariant Cohomology.

PROBLEM: If X is a space with a T-action, this action is not necessarily free. So we study a slightly modified space that does.

Consider a contractible space ET with a free T-action. (the universal bundle exists for compact Lie Groups, T is such)

Let $BT = ET/T$ the classifying space of T.

The diagonal action of $T \circlearrowright X \times ET$ is free (it is free on ET). This is the total space in the *Borel fibration*

$$\begin{tikzcd} \text{fiber}: X \arrow[r] & X \times_T ET := X \times ET/T \arrow[d] \\ & BT \ : \text{base} \end{tikzcd}$$

Ex: $X = pt$, $T = S^1$.

Notice $T = S^1$ acts on each $S^{2n+1} \overset{\text{unit sphere}}{\subseteq} \mathbb{C}^{n+1}$ by rotation.

$$\underset{\substack{\cap \\ S^1}}{e^{i\theta}} \cdot \underset{\substack{\in \\ S^{2n+1}}}{(z_0, \dots, z_n)} = (e^{i\theta} z_0, \dots, e^{i\theta} z_n)$$

But S^{2n+1} is not contractible, but if $S^{2n+1} \subseteq S^{2n+3}$, *do some topology* $\Rightarrow$ cycles with $\dim \leq 2n-2$ are contractible in S^{2n+3}.

Take $ES^1 = \bigcup_{n \geq 0} S^{2n+1} = S^\infty$. ← This is contractible

Then, $BS^1 = ES^1/S^1 = S^\infty/S^1 \cong \mathbb{CP}^\infty$.

Then, $H^*_{S^1}(pt) = H^*(pt \times_{S^1} ES^1) = H^*(ES^1/S^1)$

$= H^*(\mathbb{C}P^\infty)$.

Recall: $H^*(\mathbb{C}P^k) \cong \mathbb{C}[t]/(t^{k+1})$.

use CW-complex structure of $\mathbb{C}P^\infty$

$\longrightarrow H^*(\mathbb{C}P^\infty) \cong \mathbb{C}[t]$

* In general, how do we compute cohomology of the fiber bundle $X \times_T ET$?

Serre spectral sequence.

Equivariant Cohomology is nice because:

1. Functoriality
2. It's a ring!
3. Mayer-Vietoris
4. Künneth Formula
5. Leray (-Serre) spectral sequence
6. If X smooth, orientable: Poincaré duality
7. Chern classes exist.

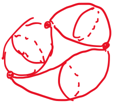
Nice fact we get for free: $(\cong \mathbb{C}[t_1,\dots,t_n])$

- $H_T^*(X)$ is a module over $H_T^*(pt)$

$X \longrightarrow pt$ $\quad H_T^*(pt) \longrightarrow H_T^*(X)$

GKM-SPACES.

The key to everything is to study the fixed points of the torus action.



← Ex. of a space that is not equivariantly formal.

Call the following the GKM Conditions:

(1) X has finitely many T-fixed points

(2) X has finitely many one-dimensional T-orbits

(3) X is equivariantly formal. (Leray-Serre spectral sequence degenerates at E_2 term)

* A variety X satisfying all 3 is called a GKM Space.

The following imply (3):

- X is smooth, complex projective variety
- X has no odd dim. cohomology
- X has a T-stable CW-decomposition

$(t_1, t_2, t_3) \cdot [x_0, x_1, x_2]$

Ex: Let $(\mathbb{C}^*)^2 \circlearrowright \mathbb{CP}^2$ by $(t_1,t_2)\cdot[x_0,x_1,x_2] = [x_0, x_1t_1, x_2t_2]$.

$= [t_1x_0, t_2x_1, t_3x_2]$

$= [(\frac{t_1}{t_3})x_0, (\frac{t_2}{t_3})x_1, x_2]$

Fixed pts: $[1,0,0]$, $[0,1,0]$, $[0,0,1]$

1D-Orbits: $[*,*,0]$, $[*,0,*]$, $[0,*,*]$ $(\cong \mathbb{CP}^1)$

$[1,a,0]$

$\downarrow$

$[1,at_1,0]$

Note the closure of each orbit has exactly two fixed points, called the north and south poles.

THEOREM. (GORESKY, KOTTWITZ, MACPHERSON).

If X is a GKM space, then

$$H_T^*(X) \cong \left\{ (p_v) \in \left(H_T^*(pt)\right)^{\oplus |X^T|} : \begin{array}{l} \text{for each 1-D orbit with} \\ \text{poles } N_0, S_0, \text{ weight} \\ \alpha,\ p_N - p_S \in \langle \alpha \rangle \end{array} \right\}$$

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ Into combinatorics land ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

MOMENT GRAPHS.

- Let X be a GKM space.
- $T' \subseteq T$ a 1-D subtorus

The moment graph of X is a labeled, directed graph G:

(1) $V(G) = X^T$

(2) $E(G) = \{ (v,w) \mid \exists$ 1-D T-orbit O with $\{v,w\} \subseteq \overline{O} \}$

(3) label of (v,w) is α ⟺ T-weight of $T_N(O)$ is $\pm\alpha$

(4) Edges are directed consistent with flow of T' on O.

Ex: $Gr_{\mathbb{C}}(2,4)$. The torus $(\mathbb{C}^*)^4 = (t_1,t_2,t_3,t_4)$ acts on $\mathbb{C}^4$ under the diagonal action:

$$(t_1,t_2,t_3,t_4)\cdot(z_1,z_2,z_3,z_4) = (t_1z_1, t_2z_2, t_3z_3, t_4z_4).$$

$Gr_{\mathbb{C}}(2,4)$ inherits this action.

What are the fixed points in $Gr_{\mathbb{C}}(2,4)$ of $(\mathbb{C}^*)^4$?

Let $\binom{n}{k} = \left\{ \lambda_1 \ldots \lambda_n \text{ strings with } k \text{ ones, } n-k \text{ zeroes} \right\}$ (the cardinality works out)

Let $\mathbb{C}^\lambda = \bigoplus_{i=1}^{n} \mathbb{C}^{\lambda_i}$. Then, the $\mathbb{C}^{\lambda'}$'s are the fixed points ($\lambda \in \binom{n}{k}$) under the $T^n = (\mathbb{C}^*)^n$ action.

For $Gr_{\mathbb{C}}(2,4)$: $\binom{4}{2} = \{0011, 0101, 1001, 0110, 1010, 1100\}$

$$0011 = \begin{array}{c} \begin{matrix} 0 & 0 & 1 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad 0110 = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 0 & 1 & 1 & 0 \end{matrix} \end{array}$$

$$0101 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 1010 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$1001 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 1100 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

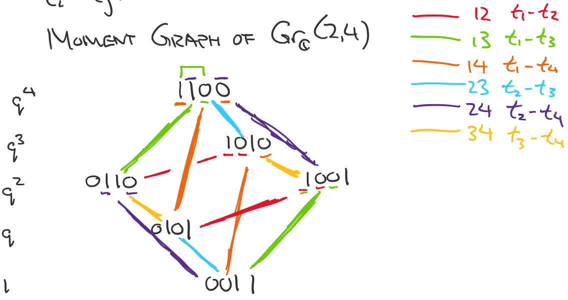
$$\begin{bmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & at_4 \\ 0 & 0 & 1 & bt_4 \end{bmatrix}$$

Well... $(\mathbb{C}^*)^4$ does nothing to these.

For the Grassmannian: fixed points $\lambda, \lambda' \in \binom{n}{k}$ are connected by an edge $\iff$ *there is a transposition* $(i\,j)$ s.t. $(i\,j)\lambda = \lambda'$.

Furthermore, (if $i<j$), the *label* on the edge (λ, λ') is $t_i - t_j$.

MOMENT GRAPH OF $Gr_{\mathbb{C}}(2,4)$

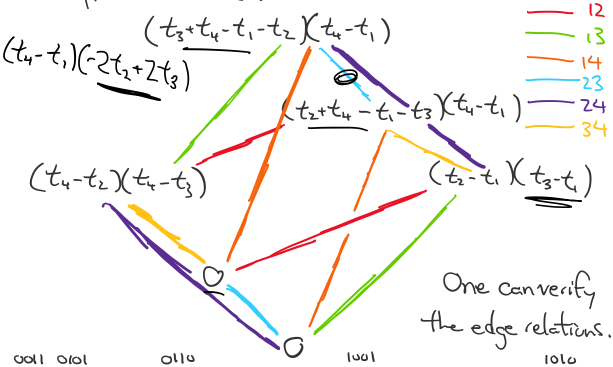


CANONICAL CLASSES. (Schubert Classes in Tao-Knutson)

Since $H_T^*(X)$ is an $H_T^*(pt)$-module, wouldn't it be nice if we had a module basis?

We use moment graphs to express elements of $H_T^*(X)$:

Example of $[c] \in H^*_{(\mathbb{C}^*)^4}(Gr_{\mathbb{C}}(2,4))$:



One can verify the edge relations.

$$\Big(\overset{\substack{0011\\ \downarrow}}{0},\ \overset{\substack{0101\\ \downarrow}}{0},\ \overset{\substack{0110\\ \downarrow}}{(t_4-t_2)(t_4-t_3)},\ \overset{\substack{1001\\ \downarrow}}{(t_2-t_1)(t_3-t_1)},\ \overset{\substack{1010\\ \downarrow}}{(t_2+t_4-t_1-t_3)(t_4-t_1)},\ \underset{\substack{\uparrow\\ 1100}}{(t_3+t_4-t_1-t_2)(t_4-t_1)}\Big)$$

CANONICAL CLASSES

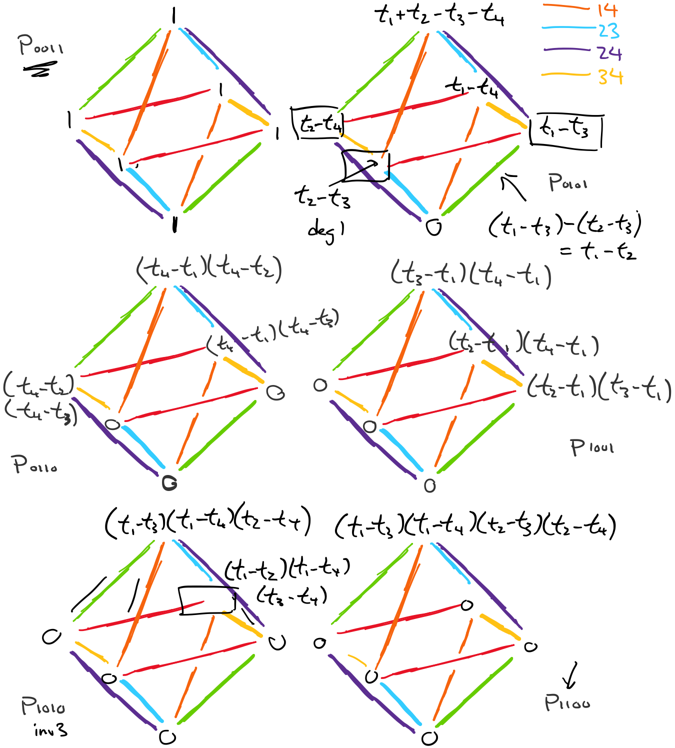
(Need certain conditions – satisfied by
- Flag varieties
- Grassmannians
- Schubert varieties)

- For each $p \in X^T$,

(1) Set $p_v(u) = 0$ if there is no chain $u \mapsto u_1 \mapsto \dots \mapsto v$ on moment graph

(2) Set $p_v(v) = \prod_{w: v \mapsto w}$ (label on edge $v \mapsto w$)

(3) Pick homogeneous polynomials of deg $p_v(v)$ that satisfy GKM edge relations.



Fun side note: We can grade these elts by $\deg_v(v)$.

For the Grassmannian $Gr_{\mathbb{C}}(k,n)$: $\deg_v(v) = \mathrm{inv}(v)$ $\left(v \in \binom{n}{k}\right)$

Where $\mathrm{inv}(v) = \#\left\{\begin{array}{l}\text{pairs } 1 \le i < j \le n \text{ with} \\ v_i = 1 \;\; v_j = 0\end{array}\right\}$

0011	0101	1001	0110	1010	1100
0	1	2	2	3	4

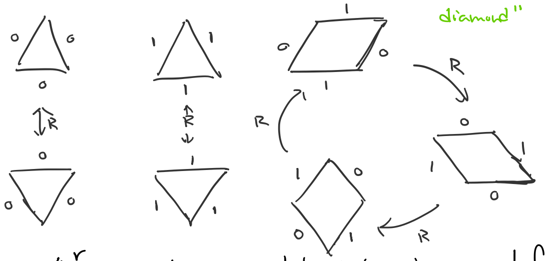
$\binom{n}{k}$ is the set of $(k, n-k)$ shuffles. So we have

$$\underbrace{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \sum_{(k,n-k)\text{ shuffles }\pi} q^{\mathrm{inv}(\pi)} = \sum_{v\in\binom{n}{k}} q^{\mathrm{inv}(v)} = \mathcal{H}(H^*(Gr_{\mathbb{C}}(n,k)))$$

(once you pass back to regular cohomology)

Puzzles.

Consider the three "puzzle pieces":



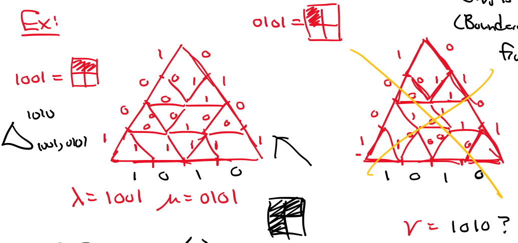
I remember it by "1 is on the left in diamond"

Call a $\Delta^{\nu}_{\lambda,\mu}$-puzzle an equilateral triangle composed of the above puzzle pieces, with boundary $\partial P = \lambda, \mu, w_0 \cdot \nu$

(w_0 is the long word)

(Boundary is read clockwise from SW corner)

Ex:



$\lambda = 1001$ $\mu = 0101$

$\nu = 1010$?

$S_2 S_1 = \ldots + (1) S_{21}$

Recall: in $H^*(Gr_{\mathbb{C}}(k,n))$, we have that

$$\sigma_\lambda \sigma_\mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} \sigma_\nu \quad \text{for fixed } \lambda,\mu \in \binom{n}{k},$$

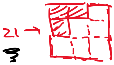
ν ranges over all $\binom{n}{k}$.

The $c^{\nu}_{\lambda,\mu}$ are the *Littlewood-Richardson coefficients*, which are the coefficients $S_\lambda S_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} S_\nu$ with S_λ the Schur function corresponding to λ.

THEOREM. (TAO, KNUTSON, WOODWARD) [Puzzles compute Schubert Calculus]. Let λ, μ, ν index Schubert classes in $H^*(Gr_{\mathbb{C}}(k,n))$

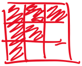
The coefficients $c^{\nu}_{\lambda,\mu}$ count $\Delta^{\nu}_{\lambda,\mu}$ puzzles.

Ex. $c^{321}_{21,21} = 2$. Why? Study $Gr_{\mathbb{C}}(3,6)$!



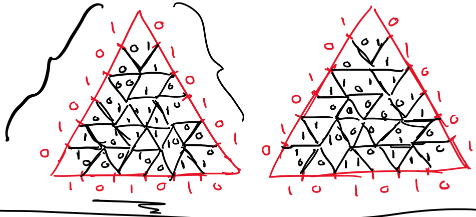
= 010101

321



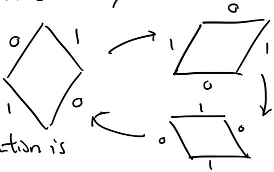
= 101010

$s_{21}s_{21} = \dots + 2s_{321} + \dots$

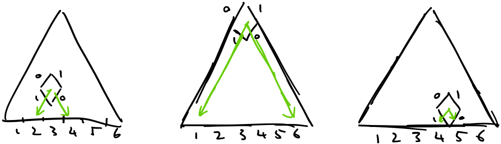


As it turns out... this extends to Equivariant cohomology.

Add the puzzle piece.

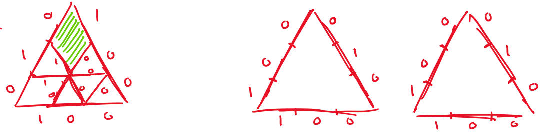


The torus weight contribution is



$wt(P) = t_4 - t_2$ $\qquad t_6 - t_1$ $\qquad t_5 - t_4$

Ex:



$$\tilde{S}_{010}\tilde{S}_{100} = (t_3-t_1)\tilde{S}_{100} \qquad S_{100}S_{010} = (t_2-t_1)\tilde{S}_{100} + (t_3-t_2)\tilde{S}_{100}$$

Equiv. classes
ariant

THEOREM. (TAO, KNUTSON)

Let $\lambda, \mu, \nu \in \binom{n}{k}$, indexing Schubert classes $\tilde{S}_\mu, \tilde{S}_\lambda, \tilde{S}_\nu$ in $Gr_{\mathbb{C}}(k,n)$. Then:

(1) $\tilde{C}^{\nu}_{\lambda,\mu}$ structure constant is the sum of the weights of all puzzles $\Delta^{\nu}_{\lambda,\mu}$.

(2) $$\tilde{S}_\lambda \tilde{S}_\mu = \sum_{\text{Puzzles } \Delta^{\nu}_{\lambda,\mu}}' wt(P)\tilde{S}_\nu$$

Next time (?) Flag Varieties & Their Subvarieties...?

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