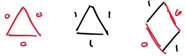
Equivariant Puzzle Pieces, Pieri Rule, Brief Sketch of Proof

Tuesday, November 17, 2020 6:15 PM

Recall: (1) The equivariant cohomology rings of "nice enough" varieties (satisfying the GKM conditions) have a nice combinatorial representation, called the GKM presentation. In particular, for $X = Gr_{\mathbb{C}}(k,n)$ and $T = (\mathbb{C}^*)^n$ acting on the Grassmannian,

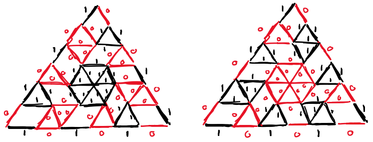
$$H_T^*(X) = \left\{ (p) \in \underset{\underset{(\mathbb{C}[t_1,\dots,t_n])^{|\binom{n}{k}|}}{\cong}}{H_T^*(pt)^{|\binom{n}{k}|}} : \begin{array}{l} t_i - t_j \,\big|\, ((p)|_\lambda - (p)|_{\lambda'}) \\ \text{if } \lambda, \lambda' \text{ differ exactly} \\ \text{at the } i,j^{th} \text{ spots} \Leftrightarrow \lambda = (i\,j)\lambda' \end{array} \right\}$$

(2) The Littlewood-Richardson coefficients $c^{\nu}_{\lambda,\mu}$ have a nice combinatorial representation in terms of *puzzles* with pieces



, up to rotation.

For example, $c^{321}_{21,21} = 2$ because the only two puzzles $\Delta^{101010}_{010101,010101}$ are given by:

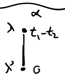


Goals for today: — Show that the puzzle construction extends to equivariant cohomology

— Brief description of Pieri rule & Equivariant Pieri Rule

— Give a brief sketch of the proof

Prelude: Multiplying in GKM Rings.



$\alpha|_\lambda = t_1 - t_2$

$\alpha|_{\lambda'} = 0.$

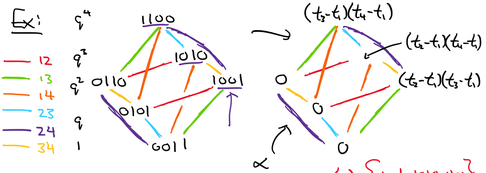
A few definitions:

- The support of a class α, denoted $\mathrm{supp}(\alpha)$ of a class α is the set $\mathrm{supp}(\alpha) = \left\{ \lambda \in \binom{[n]}{k} \,\middle|\, \alpha|_\lambda \neq 0 \right\}$
- We say that $\lambda \geq \lambda'$ for two strings λ, λ' if $\sum_{i=0}^{j} \lambda_i \geq \sum_{i=0}^{j} \lambda_i'$ for all $j = 1, 2, \dots, n$

  $\Updownarrow$

  There exists a directed path $\lambda \mapsto \dots \mapsto \lambda'$ in the moment graph of $Gr_{\mathbb{C}}(k,n)$.
- We say α is supported above λ if $\lambda' \geq \lambda \;\; \forall \lambda' \in \mathrm{supp}(\alpha)$.

Ex:



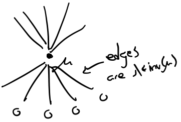
$\operatorname{supp}(\alpha) = \{1001, 1010, 1100\}$

α is supported above 1001

Proposition 1. Schubert classes $\tilde{S}_\lambda$ form an $H_T^*(pt)$-basis of $H_T^*(Gr_{\mathbb{C}}(k,n))$. Additionally, the expansion can be written uniquely using only λ with $\lambda \geq \mu$ for some $\mu \in \operatorname{supp}(\alpha)$.

Proof. Use upper-triangularity of Schubert classes.

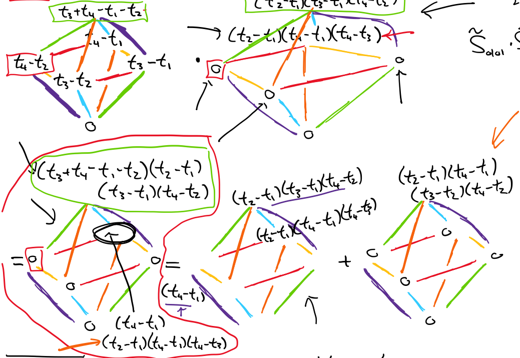
Linear independence: Suppose $\sum_\lambda' \gamma_\lambda \tilde{S}_\lambda = 0$ for some $\gamma_\lambda \in H_T^*(pt)$ not all 0. Choose $\mu \in \binom{n}{k}$ to be the element which is minimal (w.r.t. $\leq$ defined earlier) in the lattice of $\binom{n}{k}$ and $\gamma_\mu \neq 0$. But then $\sum_\lambda' \gamma_\lambda \tilde{S}_\lambda \big|_\mu \neq 0$. ⚡



$wt_\lambda \mid \alpha|_\mu - 0$

Span: Choose $\mu \in \operatorname{supp}(\alpha)$ minimal. The GKM conditions force $\prod_{(i,j) \in \operatorname{inv}(\mu)} (t_j - t_i) \mid \alpha|_\mu =: \beta$. Subtract $\beta \tilde{S}_\mu$ and repeat the argument. Notice this construction only uses $\lambda \geq \mu$ for some $\mu \in \operatorname{supp}(\alpha)$.

Ex: $\tilde{S}_{0101} \tilde{S}_{1010}$


$t_3+t_4-t_1-t_2$
t_4-t_1
t_4-t_2
t_3-t_1
t_3-t_2
0
$\cdot$
$(t_2-t_1)(t_3-t_1)(t_4-t_2)$
$(t_2-t_1)(t_4-t_1)(t_4-t_3)$
0
0
0
0
$(t_3+t_4-t_1-t_2)(t_2-t_1)$
$(t_3-t_1)(t_4-t_2)$
$=$
0
0
0
0
(t_4-t_1)
$(t_2-t_1)(t_4-t_1)(t_4-t_3)$
$=$
(t_4-t_1)
$(t_2-t_1)(t_3-t_1)(t_4-t_2)$
$(t_2-t_1)(t_4-t_1)(t_4-t_3)$
+
$(t_2-t_1)(t_4-t_1)$
$(t_3-t_2)(t_4-t_2)$
0
0
0
0
0

$\operatorname{supp}(\tilde{S}_{0101}\tilde{S}_{1010}) = \{1010, 1100\}$

$$\tilde{S}_{0101} \cdot \tilde{S}_{1010} = \Big(\underbrace{(t_4-t_1)^2(t_2-t_1)(t_4-t_3)}_{1010},\ \underbrace{(t_3+t_4-t_1-t_2)(t_2-t_1)(t_3-t_1)(t_4-t_2)}_{1100}\Big)$$

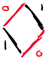
$$= (t_4-t_1)\Big(\underbrace{(t_2-t_1)(t_4-t_1)(t_4-t_3)}_{1010},\ \underbrace{(t_2-t_1)(t_3-t_1)(t_4-t_2)}_{1100}\Big)\Big\}\tilde{S}_{1010}$$

$$+ \Big(\underbrace{0}_{1010},\ \underbrace{(t_2-t_1)(t_4-t_1)(t_3-t_2)(t_4-t_2)}_{1100}\Big)\Big\}\tilde{S}_{1100}$$

As it turns out — this too is computed by puzzles.

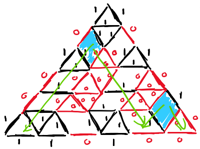
The equivariant puzzle piece is the following:

CAUTION: THIS PUZZLE PIECE CANNOT BE ROTATED.



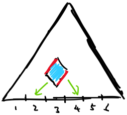
1 is on the right of the diamond

An example of a puzzle using an equivariant piece:

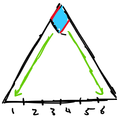


Which contributes weight $\underline{(t_5-t_1)}\underline{(t_6-t_5)}$ to the calculation of $c_{100110,101010}^{110100}$.

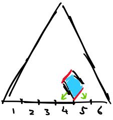
An equivariant puzzle piece p contributes a weight wt(p), computed in the following way:



$wt(p) = t_4 - t_2$



$wt(p) = t_6 - t_1$



$wt(p) = t_5 - t_4$

The weight of an equivariant puzzle P is $\underline{\prod_{p\in P} wt(p)}$.

THEOREM. (Knutson, Tao 2001). Puzzles compute equivariant cohomology.

For $0\le k\le n$, $\lambda,\mu,\nu\in\binom{n}{k}$ and Schubert classes $\tilde{S}_\lambda, \tilde{S}_\mu, \tilde{S}_\nu \in H_T^*(Gr_{\mathbb{C}}(k,n))$, then:

equivariant structure constant

(1). The structure constant $c_{\lambda,\mu}^{\nu}$ is the sum of the weights of equivariant puzzles $\Delta_{\lambda,\mu}^{\nu}$.

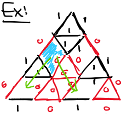
$\Updownarrow$

(2). $\tilde{S}_\lambda \tilde{S}_\mu = \sum_{\substack{\text{puzzles } P \\ \partial P = \Delta_{\lambda,\mu}^{\nu}}} wt(P)\,\tilde{S}_\nu$

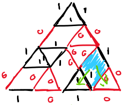
Note: Setting $t_1,\dots,t_n=0$ recovers mult. in $H^*(X)$.

Ex:

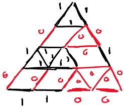
$\tilde{S}_{0101}\cdot\tilde{S}_{1010}$



$t_3 - t_1$



$t_4 - t_3$



$(t_4-t_1)\tilde{S}_{1010} + \tilde{S}_{1100}$

//

$Gr_{\mathbb{C}}(2,4)$

deg=1. deg=3

$\lambda=[1]$

$$\tilde{S}_{0101}\cdot\tilde{S}_{1010} = (t_3-t_1)\tilde{S}_{1010} + (t_4-t_3)\tilde{S}_{1010} + \tilde{S}_{1100}$$

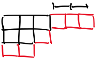
$\lambda=[2,1]$ $\lambda=[2,1]$ $\lambda=[2,1]$ $\lambda=[2;2]$

(EQUIVARIANT) PIERI RULE

We first recall the ordinary Pieri rule:

$$s_\mu h_r = \sum_{\lambda}{}' s_\lambda$$
$|\mu|+r=|\lambda|$
no two added boxes in same column

Ex: $S_{3,2} \cdot h_3 = S_{6,2} + S_{5,3} + S_{5,2,1} + S_{4,3,1} + S_{4,2,2} + S_{3,3,2}$



$d^{\nu}_{\lambda,\mu}$

Showing that a certain set of coefficients ^ satisfy Pieri rule
$\Longrightarrow$ some wizardry with recurrences $\Longrightarrow d^{\nu}_{\lambda,\mu} = c^{\nu}_{\lambda,\mu}$

"Puzzlifying" Pieri Rule (Assume $0<k<n$) (Make puzzle friendly?)

More definitions:

- $\text{div} = 0\ldots0101\ldots1$ be the unique elt. of $\binom{n}{k}$ with $\text{inv}(\text{div}) = 1$

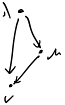
$00\ldots0\underbrace{1}\ldots1$
$\downarrow$
$0\ldots0101\ldots1$

- $\lambda' \to \lambda$ be the covering relation in the lattice $\binom{n}{k} \leftarrow$ GKM moment graph

$\Updownarrow$

$\lambda' > \lambda$ and $\ell(\lambda') = \lambda + 1 \Longleftrightarrow$ $\lambda' = \alpha\underline{10}\beta$, $\lambda = \alpha\underline{01}\beta$ for strings α, β

$\lambda \not\to \nu$
$\lambda \to \mu$
$\mu \to \nu$



Quick tidbit! (Converting between strings and λ) $\leftarrow$ Where S_λ follows Manivel's def.

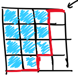
In English notation (:⌒), we have the following bijection between strings and partitions in an $k \times (n-k)$ box:

(1) Consider lattice paths $(k, n-k) \to (0,0)$. (SW lattice paths)

(2) Let 0 be a west step, 1 be a south step.

$S_\lambda = \{ W \in G_{\mathbb{C}}(k,n)$
$\dim(W \cap V_j) = i$
if $n-k+i-\lambda_i \le j \le n-k+i-\lambda_{i+1} \}$

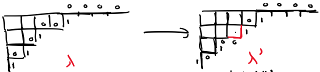
$\underline{01110100} \longrightarrow$



$\Longrightarrow \lambda = (3,3,3,2)$

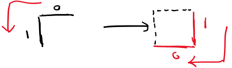
Or, write the inversions each 1 gives in order.

We use this bijection to "puzzlify" Pieri rule:



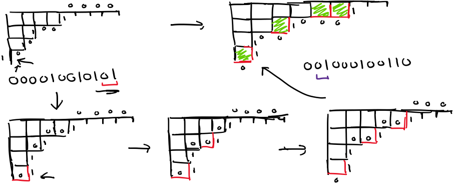
$0000100\underline{01}0101 \longrightarrow 000010\underline{10}0101$

In particular, we make the following substitution:



$01 \to 10$ to avoid conflicts

For multiple boxes, we can add the boxes left to right:



$\underset{}{000010010\underbrace{1}\underbrace{10}}$ $\quad$ $\underset{}{00001\underbrace{0}1\underbrace{0}00110}$ $\quad$ $\underset{}{00\underbrace{1}\underbrace{0}0100110}$

So that if λ' is obtained from λ by adding r boxes

$\Downarrow$

We perform $01 \mapsto 10$ r times right to left

$\Downarrow$

We multiplied λ by r transpositions $(i\ i{+}1)$, each adding one inversion (in order)

$\Downarrow$

$\lambda' = (i_r\ i_r{+}1)\ldots(i_1\ i_1{+}1)\lambda$ for an appropriate choice of $\underline{i_1 > i_2 > \ldots > i_r}$

$(\Uparrow)$

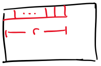
$\Downarrow$

$\lambda' \geq \lambda$ in the $\binom{n}{k}$ lattice. (Recall the GKM moment graph), and $\underline{inv(\lambda') = inv(\lambda) + r}$.

So we can rewrite the Pieri Rule in puzzle-friendly form:

$$S_r S_\lambda = \sum'_{\substack{\lambda' \geq \lambda \\ \lambda' = (i_r\ i_r+1)\ldots(i_1\ i_1+1)\lambda,\ i_1 > \ldots > i_r}} S_{\lambda'}$$

(abuse notation: r is the string corresponding to



$\underbrace{0\ldots0}_{n-k-r}\underbrace{10\ldots0}_{r}\underbrace{1\ldots1}_{k-1}$

Specializing to $S_r = S_1 = S_{div}$,
notice $S_{\lambda'} \geq S_\lambda$ and $inv(\lambda') = inv(\lambda) + 1$
$\Longrightarrow S_{\lambda'} \to S_\lambda$, so

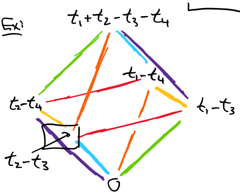
$$S_{div} S_\lambda = \sum'_{\lambda' \to \lambda} S_{\lambda'}.$$

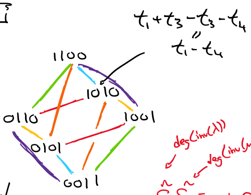
Equivariant Pieri Rule.

Lemma 3. $\tilde{S}_{div}\Big|_\lambda := \sum_{j=1}^{n} \lambda_j t_j - \sum_{i=k+1^*}^{n} t_i$

* This is incorrectly written as "1" in the paper

Ex:





$\swarrow$

Lemma 2. (1) $\boxed{c^{\nu}_{\lambda,\mu}}$ has degree $inv(\lambda) + inv(\mu) - inv(\nu)$

$\overset{deg(inv(\lambda))}{\tilde{S}_\lambda}\ \overset{deg(inv(\mu))}{\tilde{S}_\mu} = \ldots + c^{\nu}_{\lambda\mu} \tilde{S}_\nu$

$deg(inv(\lambda) + inv(\mu))$

(2) $\underline{c^{\nu}_{\lambda,\mu} = 0}$ unless $\underline{\nu \geq \lambda, \mu}$ and $\underline{inv(\nu) \leq inv(\lambda) + inv(\mu)}$.

(3) When $inv(\nu) = inv(\lambda) + inv(\mu)$, $\underline{c^{\nu}_{\lambda,\mu}}$ are integers, and agree with those from ordinary cohomology.

(4) If $\lambda = \nu$, then $c^{\lambda}_{\lambda,\mu} = \tilde{S}_\mu|_\lambda$.

$\nu < \lambda$

$$\tilde{S}_\mu\Big|_\lambda \tilde{S}_\lambda\Big|_\lambda = c^{\lambda}_{\lambda\mu} S_\lambda\Big|_\lambda + \underline{\sum_{\nu : \nu \neq \lambda} c^{\nu}_{\lambda\mu} \tilde{S}_\nu\Big|_\lambda}$$

$\alpha \in supp(\nu)$ min $\quad$ $\beta \in supp(\lambda)$ minimal

$\alpha < \beta$

Finally... equivariant Pieri rule.

$$\tilde{S}_{div}\tilde{S}_\lambda = \underline{\left(\tilde{S}_{div}|_\lambda\right)\tilde{S}_\lambda + \sum_{\lambda'\to\lambda}{}' \tilde{S}_{\lambda'}}.$$

Exercise: Check this holds for the example we did earlier,

$$\tilde{S}_{0101}\tilde{S}_{1010}.$$

PROOF. Repeated use of lemma 2. First, we can say

$$\tilde{S}_{div}\tilde{S}_\lambda = \left(\tilde{S}_{div}|_\lambda\right)\tilde{S}_\lambda + \underbrace{\sum_{\lambda'\geq\lambda}{}' c^{\lambda'}_{div,\lambda}\underbrace{\tilde{S}_{\lambda'}}_{}}_{}$$

Since in the sum, we must have:
- $c^{\lambda'}_{div,\lambda}$ has degree $1+\ell(\lambda)-\ell(\lambda')$

$$\underline{\ell(\lambda')\leq 1+\ell(\lambda)}$$

So $\ell(\lambda')=\ell(\lambda)$ or $\ell(\lambda)+1$.

But since $\ell(div)=1$, it must be $\ell(\lambda)+1$, so all nonzero $c^{\lambda'}_{div,\lambda}$ must have degree 0. We then obtain:

$$\tilde{S}_{div}\tilde{S}_\lambda = \left(\tilde{S}_{div}|_\lambda\right)\tilde{S}_\lambda + \sum_{\lambda'\to\lambda} c^{\lambda'}_{div,\lambda}\tilde{S}_{\lambda'}.$$

The rest follows from the ordinary Pieri rule.

(The image of $(\tilde{S}_{div}|_\lambda)\tilde{S}_\lambda$ under the forgetful map $H^*_T(X)\to H^*(X)$ is 0)

THEOREM. (Molev, Sagan; Okounkov) We have the recurrence:

$$\left(\tilde{S}_{div}|_\nu - \tilde{S}_{div}|_\lambda\right)c^\nu_{\lambda\mu} = \left(\sum_{\lambda'\to\lambda}{}' c^\nu_{\lambda'\mu} - \sum_{\nu'\to\nu}{}' c^{\nu'}_{\lambda\mu}\right)$$

PROOF: Expand $(\tilde{S}_{div}S_\lambda)S_\mu$ and $\tilde{S}_{div}(S_\lambda S_\mu)$ with equivariant Pieri rule; associativity implies they're equal. Compare coefficients.

$$S_\lambda = \sum_{T\in SSYT(\lambda)}{}' x^T$$

Corollary: (Conditions Puzzles Need to Satisfy)

If we have a map $\underset{\binom{n}{k}^3}{\overset{}{(\lambda,\mu,\nu)}} \mapsto \underset{H^*_T(pt)}{d^\nu_{\lambda,\mu}}$ such that:

(1) $\forall\lambda\in\binom{n}{k}$, $d^\lambda_{\lambda,\lambda} = \prod_{(i,j)\in inv(\lambda)}(t_j-t_i)$

(2) $\forall\lambda,\mu\in\binom{n}{k}$, $\left(\tilde{S}_{div}|_\lambda - \tilde{S}_{div}|_\mu\right)d^\lambda_{\lambda\underline{\mu}} = \sum_{\mu'\to\mu}{}' d^\lambda_{\lambda\mu'}$

(3) $\forall\lambda,\mu,\nu\in\binom{n}{k}$,

$$\left(\tilde{S}_{div}|_\nu - \tilde{S}_{div}|_\lambda\right)d^\nu_{\lambda\mu} = \sum_{\lambda'\to\lambda}{}' d^\nu_{\lambda'\mu} - \sum_{\nu'\to\nu}{}' d^{\nu'}_{\lambda,\mu}$$

Then $c^\nu_{\lambda,\mu} = d^\nu_{\lambda,\mu}$ $\forall\lambda,\mu,\nu$.

Use (1), (2) to show $d^\lambda_{\lambda,\mu} = \tilde{S}_\mu|_\lambda \overset{\text{Lemma 2}}{=} c^\lambda_{\lambda,\mu}$ by inducting on $\ell(\lambda)-\ell(\mu)$ with base case $\lambda=\mu$. Then, do the case for $d^{\nu}_{\lambda,\mu}$ by inducting on $\ell(\nu)-\ell(\lambda)$. The base case $\nu=\lambda$ follows from lemma 2 and prev.

DUALIZING. The dual of a puzzle is obtained by:

$$\begin{tikzcd}
P \arrow[r, mapsto] \arrow[d, "\partial"'] & P^* \arrow[d, "\partial"] \\
\Delta^{\nu}_{\lambda,\mu} & \Delta^{\nu^*}_{\mu^*,\lambda^*}
\end{tikzcd}$$

(1) Reflect

(2) Switch all the 1's and 0's.

Exercise: Verify that this is still a puzzle, and work out what $wt(P^*)$ should be. ← in the ordinary case as well

Lemma: If $\{d^{\nu}_{\lambda,\mu}\}$ satisfies conditions (1), (3), and

(4) $d^{\nu}_{\lambda,\mu} = \overline{d^{\nu^*}_{\mu^*,\lambda^*}}$ (def. by $\overline{y_i} = -y_{n+1-i}$).

Also, assume $d^{\nu}_{\lambda,\mu} = 0$ unless $\nu \geq \lambda,\mu$.

Then condition (2) follows.

Compute directly, use lemma 3.

This essentially condition (2) with a slightly more restrictive one.

Knutson, Tao showed that $d^{\nu}_{\lambda,\mu} := \sum_{\substack{P \\ \partial P = \Delta^{\nu}_{\lambda,\mu}}} wt(P)$ given by equivariant puzzles satisfied (1), (3), (4).

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A. Knutson, T. Tao. *Puzzles and (Equivariant) Cohomology of Grassmannians*. 2001. arXiv:math/0112150v1

L. Manivel. *Fonctions Symétriques, Polynômes de Schubert et Lieux de Dégénérescence*. 1998. Société Mathématique de France. (Translated by John R. Swallow)