2020年11月6日 15:07

Pieri's rule, Giambelli's rule

Review:

Grassmann: Collection of all the k-dimensional vector subspaces of a vector space of dimension n.

Denote by $G(k,n)$.

Flag: A set of nested vector subspaces of dimension n.

$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$. $\quad F_\bullet$

$F_i = i$

Def Schubert varieties:

For every non-increasing partition $\iff$ Schubert varieties

$\lambda_1, \dots \lambda_k$ $\quad\quad \Sigma_{\lambda_1,\dots\lambda_k}$

$\lambda_1 \geq \lambda_2, \dots \geq \lambda_k$

$\boxed{\lambda_1 \leq n-k}$

$\Sigma_{\lambda_1,\dots\lambda_k} = \{ \Lambda \in G(k,n) \mid \dim(\Lambda \cap \underline{F_{n-k+i-\lambda_i}}) \geq i \}$

e.g. $\lambda_1 = \cdots \lambda_k = 0$.

$\Sigma_{0,\dots 0} = \{ \Lambda \in G(k,n) \mid \dim(\Lambda \cap \underline{F_{n-k+i}}) \geq i \}$

Λ vector space of dim k.

F_{n-k+i} has dimension $n-k+i$.

$\Lambda \cap F_{n-k+i}$ has dimension $(n-k+i)+k-n = i$

$\Sigma_{0,\dots,0} = G(k,n)$

e.g. $\lambda_1 > 0$, $\lambda_2 = \cdots = \lambda_k = 0$

e.g. $\lambda_1 > 0, \lambda_2 = \cdots = \lambda_k = 0$

$\dim(\Omega \cap F_{n-k+1-\lambda_1}) \geq 1$

$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-k+1-\lambda_1} \subseteq F_{n-k+1-\lambda_1+1} \subseteq \cdots$

$\subseteq F_{n-k+1} \subseteq \cdots\cdots$

$\uparrow$

Ω automatically intersect this and have dimension at least 1

What is $\Sigma_{\lambda_1,0\ldots0}$?

$\Sigma_{\lambda_1,0\ldots0}$ = all the k-dimensional vector ^sub^ spaces such that it intersect a given vector subspace of dimension $n-k+1-\lambda_1$ with dimension of intersection to be at least 1.

$\Sigma_{1,0}$ in $G(2,4)$.

Denote the Schubert cycles by $\sigma_{\lambda_1,\ldots\lambda_k}$.

Def Pieri's cycle / special Schubert cycle:

$\sigma_{\lambda_1,0,0\ldots} = \sigma_{\lambda_1} \qquad \sigma_\lambda = \sigma_{a,0,0\ldots}$

Def weight of partition $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$

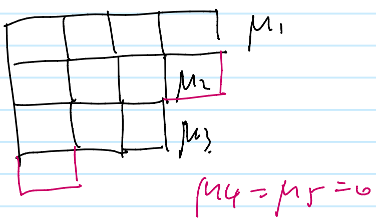
Pieri's rule: σ_λ is a pieri's cycle.

σ_μ is a general schubert cycle. $\mu = \mu_1, \ldots \mu_k$

$\sigma_\lambda \cdot \sigma_\mu = \sum \sigma_\nu \quad (\nu = \nu_1, \ldots, \nu_k)$

$|\nu| = |\lambda| + |\mu|$

$\mu_i \leq \nu_i \leq \mu_{i-1}$

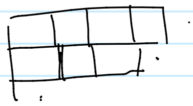


$\underset{0}{\mu_5} \leq \nu_5 \leq \overset{0}{\mu_4}$

$\underset{3}{\mu_4} \leq \nu_4 \leq \mu_3$

3.

Careful: $\lambda_1 \le n-k$



e.g. $\sigma_2 \cdot \sigma_{4,3} = \sigma_{6,3} + \sigma_{5,4} + \sigma_{4,3,2}$
$+ \sigma_{4,4,1} + \sigma_{5,3,1}$.

e.g. $\sigma_2 \cdot \sigma_2$ in $G(2,4)$

$= \sigma_{2,2}$.

Giambelli's rule:

$$\sigma_{\lambda_1,\dots,\lambda_k} = \begin{vmatrix} \sigma_{\lambda_1}, & \sigma_{\lambda_1+1}, & \sigma_{\lambda_1+2}, & \cdots \\ \sigma_{\lambda_2-1}, & \sigma_{\lambda_2}, & \sigma_{\lambda_2+1}, & \cdots \\ & & \ddots & \\ \cdots & & \sigma_{\lambda_{k-1}} & \sigma_K \end{vmatrix}$$

$$\sigma_{4,3,1} = \begin{vmatrix} \sigma_4 & \sigma_5 & \sigma_6 \\ \sigma_2 & \sigma_3 & \sigma_4 \\ \underset{\substack{\| \\ 0}}{\sigma_{-1}} & \underset{\substack{\| \\ 1}}{\sigma_0} & \sigma_1 \end{vmatrix} \qquad \sigma_{4+2} \quad \sigma_4{+2}$$

$\sigma_\lambda = 0$ when $\lambda < 0$

$\sigma_\lambda = 1$ when $\lambda = 0$

$$\sigma_{\lambda_1,\dots,\lambda_k} = \begin{vmatrix} \sigma_{\lambda_1} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \ddots & \sigma_{\lambda_2+k-2} \\ \sigma_{\lambda_3-2} & & \vdots \\ \vdots & & \sigma_{\lambda_k} \\ \sigma_{\lambda_k-k+1} & \cdots & \end{vmatrix}$$

$$= \sigma_{\lambda_1+k-1} \cdot \begin{vmatrix} \sigma_{\lambda_2-1} & & \\ & \ddots & \\ & & \sigma_{\lambda_k-1} \end{vmatrix} + \sigma_{\lambda_2+k-2} \cdot \begin{vmatrix} \sigma_{\lambda_1} & & & \\ & \sigma_{\lambda_3-1} & & \\ & & \ddots & \\ & & & \sigma_{\lambda_{k-1}} \end{vmatrix}$$

$$+ \cdots + \sigma_{\lambda_k} \cdot \begin{vmatrix} \sigma_{\lambda_1} & & \\ & \ddots & \\ & & \sigma_{\lambda_{k-1}} \end{vmatrix}$$

$$\Big| \quad \sigma_{\lambda_{k-1}} \Big|$$

$$= \sigma_{\lambda_1+k-1} \cdot \sigma_{\lambda_2-1,\dots,\lambda_k-1} + \sigma_{\lambda_2+k-2} \cdot \sigma_{\lambda_1,\lambda_3-1,\dots,\lambda_k-1}$$

$$+ \dots + \sigma_{\lambda_k} \cdot \sigma_{\lambda_1,\dots,\lambda_{k-1}}$$

$$=$$

$$\sigma_{4,3} = \begin{vmatrix} \sigma_4 & \sigma_5 \\ \sigma_2 & \sigma_3 \end{vmatrix} = \sigma_4 \cdot \sigma_3 - \sigma_2 \cdot \sigma_5$$

$$= \sigma_7 + \sigma_{6,1} + \sigma_{5,2} + \sigma_{4,3}$$

$$- (\sigma_7 + \sigma_{6,1} + \sigma_{5,2})$$

$$= \sigma_{4,3}$$

Map from symmetric function / schur functions } professor Carlsson's comment.

schur poly $\longrightarrow$ schubert cells