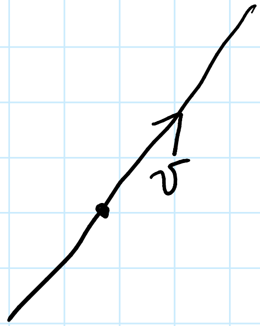
Meeting 10/2

Friday, October 2, 2020 2:45 PM

Projective space $\mathbb{CP}^n =$

= space of lines in $\mathbb{C}^{n+1}$ through the origin (can consider $\mathbb{RP}^n$, $\mathbb{F}P^n$ for a field $\mathbb{F}$)



Choose a vector v (nonzero)

$v = (z_0, z_1, \dots z_n)$ [not all $z_i = 0$]

Two vectors define same line if they are proportional: $(z_0, z_1, \dots z_n) \sim (\lambda z_0, \lambda z_1, \dots \lambda z_n)$

$[z_0 : z_1 : \dots : z_n]$ ← homogeneous coordinates. $\lambda \neq 0$

Ex $\mathbb{CP}^1$ $(z_0, z_1) \sim (\lambda z_0, \lambda z_1)$

2 cases: $z_0 \neq 0 \longrightarrow (z_0, z_1) \sim (1, \underbrace{\overset{z}{\frac{z_1}{z_0}}}_{\text{any number}}) = \mathbb{C}$

$z_0 = 0$ then $z_1 \neq 0 \Rightarrow (0, z_1) \sim (0, 1) =$ point $= \{\infty\}$

$z_0 \to 0 \quad \frac{z_1}{z_0} \to \infty$

$\mathbb{CP}^1 = \mathbb{C} \cup \underbrace{\{\infty\}}_{\text{infinite pt}}$

$\boxed{\text{Exercise Topologically } \mathbb{CP}^1 \approx S^2,\ \mathbb{RP}^1 \approx S^1}$

In general: $n+1$ cases:

• $z_0 \neq 0 \Rightarrow (z_0 : \dots z_n) \sim (1, \frac{z_1}{\ }, \frac{z_2}{\ }, \dots \frac{z_n}{\ }) \backslash \mathbb{C}^n$

- $z_0 \neq 0 \Rightarrow (z_0, \dots z_n) \sim \left(1, \frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots \frac{z_n}{z_0}\right)$ $\mathbb{C}^n$
- $z_0 = 0, z_1 \neq 0 \Rightarrow (0, z_1, \dots z_n) \sim \left(0, 1, \frac{z_2}{z_1}, \frac{z_3}{z_1} \dots \frac{z_n}{z_1}\right)$ $\mathbb{C}^{n-1}$

  ⋮

- $z_0 = \dots = z_{n-1} = 0, z_n \neq 0$ $(0, \dots 0, z_n) \sim (0, \dots 0, 1)$ pt

$$\mathbb{C}P^n = \mathbb{C}^n \sqcup \underbrace{\mathbb{C}^{n-1} \cup \mathbb{C}^{n-2} \sqcup \dots \cup pt}_{z_0 = 0 \quad \mathbb{C}P^{n-1}} \leftarrow \text{cell decomposition}$$

Ex $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1$

2d plane $\quad \{z_0 = 0\}$ = "line at infinity"

$\mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1$

Open charts $\{z_i \neq 0\} = U_i$

$U_0, U_1, \dots U_n$ cover $\mathbb{C}P^n$

$$U_i = \{(z_0, \dots z_n)\}_{z_i \neq 0} = \left(\frac{z_0}{z_i}, \dots \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \dots \frac{z_n}{z_i}\right)$$

n different independent variables in U_i

$U_i \simeq \mathbb{C}^n$

$\mathbb{C}P^n$ locally looks like $\mathbb{C}^n$, each open chart $U_i = \mathbb{C}^n$.

Smooth complex manifold of dimension n.

Ex $\mathbb{C}P^1$ has two charts:

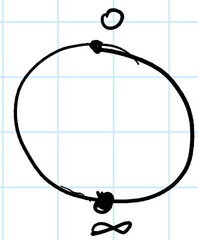
U_0

Ex $\mathbb{C}P^1$ has two charts:

$\{z_0 \neq 0\} \rightsquigarrow (z_0, z_1) \sim \left(1, \frac{z_1}{z_0} = z\right) = U_0$

$\{z_1 \neq 0\} \rightsquigarrow (z_0, z_1) \sim \left(\frac{z_0}{z_1} = w, 1\right) = U_1$ $\quad w = \frac{1}{z}$ where both are defined.

Exercise $\mathbb{C}P^1_z$



$U_0 = \{z \neq \infty\}$ = sphere minus south pole

$U_1 = \{z \neq 0\}$ = sphere minus north pole.

$\mathbb{C}P^2$ $\quad U_0 = \{z_0 \neq 0\}$ $\quad \left(1, \underset{x_1}{\frac{z_1}{z_0}}, \underset{x_2}{\frac{z_2}{z_0}}\right)$

$U_1 = \{z_1 \neq 0\}$ $\quad \left(\underset{y_1}{\frac{z_0}{z_1}}, 1, \underset{y_2}{\frac{z_2}{z_1}}\right)$

What is the relation between (x_1, x_2) and (y_1, y_2)?

$$\boxed{y_1 = \frac{1}{x_1} \quad y_2 = \frac{x_2}{x_1}}$$

Change of variables between the charts defined on $U_0 \cap U_1$

Grassmannians $G(k,n)$ = space of k-dimensional linear subspaces in $\mathbb{C}^n$

Ex: $G(1,n) = \mathbb{C}P^{n-1}$

Exercise (a) $G(k,n) = G(n-k, n)$

(b) $G(2,n)$ = space of lines in $\mathbb{C}P^{n-1}$

How to find coordinates? V = k-dim subspace

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Choose a basis $v_1, \dots v_k$, write them in a matrix

$$A = k\left\{\begin{pmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \\ & \vdots & \\ \text{---} & v_k & \text{---} \end{pmatrix}\right. \qquad \begin{array}{l} k \times n \text{ matrix} \\ \operatorname{rank}(A) = k \end{array}$$

$\underbrace{\qquad\qquad}_{n}$

Change of a basis $\Leftrightarrow$ multiplication by an invertible $k \times k$ matrix on the left.

$\boxed{A \sim G \cdot A}$, if G is a $k \times k$ matrix $\det(G) \neq 0$.

$\operatorname{rk}(A) = k \Leftrightarrow$ there is a nonzero $k \times k$ minor

Choose k columns $\underline{I} = \{i_1, \dots i_k\}$

$A_{\underline{I}}$ = $k \times k$ submatrix with these columns. $\det(A_{\underline{I}}) \neq 0$

$A \sim A_{\underline{I}}^{-1} \cdot A \leftarrow$ has identity matrix in rows $\underline{I}$

<u>Conclusion</u> For each $\underline{I} \subset \{1, \dots n\}$ k-element subset

$U_{\underline{I}}$ = open chart = $\{$all matrices A such that $\det(A_{\underline{I}}) \neq 0\}$

has coordinates = all "interesting" entries of $A_{\underline{I}}^{-1} \cdot A$ $\quad = k \cdot (n-k)$

has coordinates = all interesting entries of $A_I^{-1} \cdot A$. $= k \cdot (n-k)$

columns not in I

$$\Rightarrow \boxed{U_I = \mathbb{C}^{k(n-k)}}$$

$Gr(k,n)$ is a smooth complex manifold of dimension $k(n-k)$

<u>Ex</u> $G(2,4)$ $I = \{1,2\}$

$$A = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & \begin{matrix} * & * \\ * & * \end{matrix} \end{pmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \end{array} \qquad U_I = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(A_I)$$

row reduction

$$A_I^{-1} \cdot A = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$ reduced row echelon form.

$\leftarrow$ coordinates on the chart U_I.

<u>Cell decomposition</u> = describe cells by rref:

$$\underset{\mathbb{C}^4 = U_{12}}{\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}} \quad \underset{\mathbb{C}^3}{\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}} \quad \underset{\mathbb{C}^2}{\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{\mathbb{C}^2}{\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}} \quad \underset{\mathbb{C}}{\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \underset{pt}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Any 2×4 matrix of rank 2 can be uniquely transformed to one of these by row transforma...

<u>Schubert cells</u>

Schubert cells

So one of these by row transformation.

In general, there are $\binom{n}{k}$ Schubert cells parametrised by $I \subset \{1, \dots n\}$ = positions of 1's in rref.

Exercise*: Given $I \subset \{1, \dots n\}$ k-element subset, find the dimension of the corresponding Schubert cell.

[how many free slots in rref are there?]

Plucker coordinates: $\det(A_I)$ = all minors of A ($k \times k$).

Lots of interesting relations between them.

Adjacency of cells

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \sim \begin{pmatrix} 1 & * & 0 & * \\ \varepsilon & \varepsilon & 1 & * \end{pmatrix}$$

∞

multiply by A_{13}^{-1}

Schubert variety = closure of a Schubert cell.