The Complex Grassmanian and Kleinman's Transversality Thm

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Lemma Pt. 1

(Kleinman 1974)

$$\begin{tikzcd}
& W \arrow[dl, "p"'] \arrow[dr, "q"] & & Z \arrow[dl, "r"'] \\
S & & X &
\end{tikzcd}$$

q flat $\Rightarrow$ $\exists$ a dense open subset of S s.t.

$p^{-1}(s) \times_X Z$ is

(i) empty

(ii) equidimensional and

$$\dim p^{-1}(s) \times_X Z = \dim p^{-1}(s) + \dim Z - \dim X$$

Set Up

G := integral algebraic group scheme over an algebraically closed field

X := integral algebraic scheme with transitive G action

$$\left.\begin{array}{l} f : Y \longrightarrow X \\ f' : Z \longrightarrow X \end{array}\right\} \text{maps of integral algebraic schemes}$$

g := rational elt of G

g^Y := X-scheme given by $y \mapsto g f(y)$

$G \times Y$ is integral $\Rightarrow$ q is flat $\Rightarrow$ apply lemma!

$$\begin{tikzcd} & G\times Y \arrow[dl, "p"'] \arrow[dr, "q"] & Y \arrow[d, "f"] & Z \arrow[dl, "r"] \\ G & & X & \end{tikzcd}$$

$$q(g,y) = g\,f(y)$$

q flat $\Rightarrow$ the fibers of q are equidimensional

$$\dim\{\text{fibers of } q\} = \dim G\times Y - \dim X$$

$$\Rightarrow \quad G\times Y \times_X Z \longrightarrow Z \quad \text{is flat}$$

$$\dim G\times Y\times_X Z = \dim G\times Y + \dim Z - \dim X$$

Kleinman's Transversality thm, pt. 1, 1974

$\exists$ a dense open subset $\mathcal{U} \subseteq G$ s.t. $\forall\, g \in \mathcal{U}$, $gY \times_X Z$ is

(i) empty

(ii) equidimensional and

$$\dim gY \times_X Z = \dim gY + \dim Z - \dim X$$

Lemma Pt. 2

(Kleinman 1974)

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & & Z \arrow[dl, "r"'] \\ S & & X & \end{tikzcd}$$

- the field is characteristic 0

  and

- Z regular
- q has regular fibers

$\Rightarrow$ $\exists$ a dense open subset of S s.t.

$$p^{-1}(s) \times_X Z$$

is regular

Kleinman's Transversality thm, pt. 2, 1974

Y, Z regular

$$\Downarrow$$

$\exists$ dense open subset $\mathcal{U}' \subseteq G$

s.t. $\forall\, g \in \mathcal{U}'$,

$$gY \times_X Z \text{ is regular}$$

This only holds in Characteristic 0.

⚠ This only holds in Characteristic 0

Why?

In characteristic 0, the differential of the map

$$G \longrightarrow X$$
$$g \longmapsto gx$$

surjects $\forall$ rational $g \in G$.

$\Rightarrow$ $q : G \times Y \longrightarrow X$ is surjective at each rational point.

$\Rightarrow$ q is smooth.

Proof-ish of KTT pt. 2

- $\Bbbk$ is algebraically closed & of characteristic 0
- G is regular (reduced & homogeneous)

$\Rightarrow$ $G \times Y$ regular

$\Rightarrow$ the generic fiber of q is regular

$\Rightarrow$ the generic fiber of q is geometrically regular

$\Rightarrow$ the fibers of q over the pts in a dense open subset of X are geometrically regular.

- $G \curvearrowright X$ transitively
- q is a homogeneous map

$\Rightarrow$ all fibers of q are geometrically regular

$\square$

Summary of Kleinman's Transversality theorem

Pt. 1

$\exists$ a dense open subset $\mathcal{U} \subseteq G$ s.t. $\forall\ g \in \mathcal{U}$, $gY \times_X Z$ is

(i) empty (ii) equidimensional and

$$\dim gY \times_X Z = \dim gY + \dim Z - \dim X$$

Pt. 2

- Y, Z regular
- characteristic of the base field is 0

$\Rightarrow$ $\exists$ dense open subset $\mathcal{U}' \subseteq G$ s.t. $\forall\ g \in \mathcal{U}'$,

$gY \times_X Z$ is regular

Q: How does this help us?

$X = G(k,n)$ $\qquad$ $f : Y \hookrightarrow G(k,n)$

$G = GL(n)$ $\qquad$ $f' : Z \hookrightarrow G(k,n)$

Pt. 1 $\Rightarrow$ $gY \cap Z$ is either (i) empty

(ii) a proper intersection

Pt. 2 $\Rightarrow$ If Y, Z smooth,
then $gY \cap Z$ is smooth.

and

The intersection is transverse.

Punchline:

KTT tells us when the intersection of Schubert varieties is transverse.

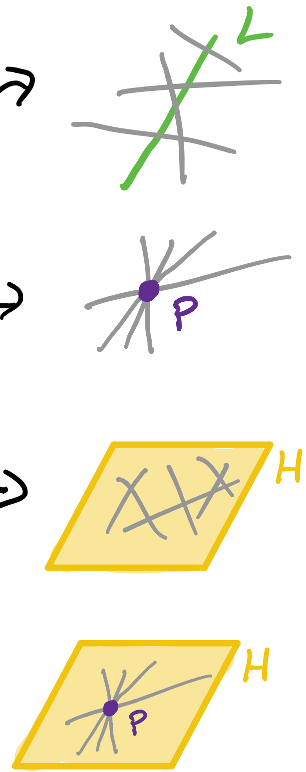
$\Rightarrow$ We can compute Littlewood-Richardson coefficients by working dually.

$\Rightarrow$ We understand products of Schubert cycles in the cohomology.

Example: $G_2(2,4) = \mathbb{G}(1,3)$

$F_\bullet : \emptyset \subset p \subset L \subset H \subset \mathbb{P}^3$

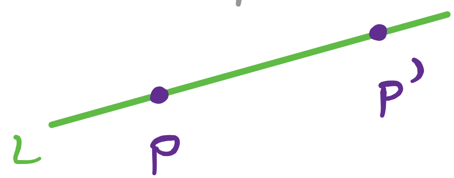
Schubert cycles	$\{\Lambda : \text{—}\}$	cycles in $\mathbb{P}^3$
$\Sigma_{0,0}$	$\mathbb{G}(1,3)$	
Σ_1	$\Lambda \cap L \neq \emptyset$	
Σ_2	$p \in \Lambda$	
$\Sigma_{1,1}$	$\Lambda \subset H$	
$\Sigma_{2,1}$	$p \in \Lambda \subset H$	
$\Sigma_{2,2}$	$\Lambda = L$	



Littlewood-Richardson coefficient $c^{\gamma}_{\lambda,\mu} = c^{2,2}_{2,2}$

$$\sigma_2^{\,2} = \overbrace{\#(\Sigma_2 \cap \Sigma_2')}^{\text{Littlewood-Richardson coefficient}} \cdot \sigma_{2,2} = 1 \cdot \sigma_{2,2}.$$

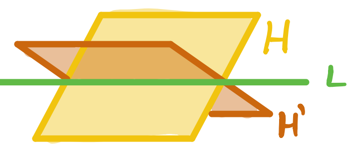
$\|$



L unique!

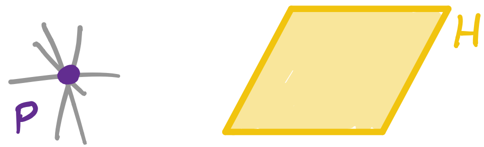
$$\sigma_{1,1}^{\,2} = \#\big(\underbrace{\Sigma_{1,1} \cap \Sigma_{1,1}'}_{\|}\big) \cdot \sigma_{2,2} = 1 \cdot \sigma_{2,2}$$

L unique!



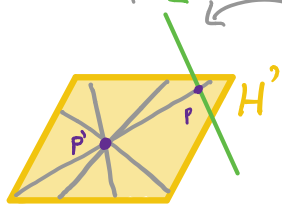
$$\sigma_2 \cdot \sigma_{1,1} = 0 \qquad \Sigma_2 \cap \Sigma_{1,1} = \emptyset$$

$$\parallel$$



$$\sigma_1 \sigma_{2,1} = \# \underbrace{\left(\Sigma_1 \cap \Sigma'_{2,1} \right)}_{= L} \cdot \sigma_{2,2} = 1 \cdot \sigma_{2,2}$$

L unique!



$$\parallel$$

$$\{ \Lambda : \Lambda \cap L \neq \emptyset \ \& \ p' \in \Lambda \subset H' \}$$

$$\sigma_1 \cdot \sigma_2 = \#(\Sigma_1 \cap \Sigma_2') \cdot \sigma_{2,1} = \sigma_{2,1}$$

$$\|$$

$$\{\Lambda : \Lambda \cap L \neq \emptyset \ \& \ p' \in L \subset H\} = \Sigma_{2,1}$$

$$\sigma_1 \cdot \sigma_{1,1} = \#(\Sigma_1 \cap \Sigma_{1,1}') \cdot \sigma_{2,1} = 1 \cdot \sigma_{2,1}$$

$$\|$$

$$\{\Lambda : L \cap H' \in \Lambda \subset H'\} = \Sigma_{2,1}$$

So far

$$\sigma_2^2 = \sigma_{1,1}^2 = \sigma_1 \sigma_{2,1} = \sigma_{2,2}$$

$$\sigma_1 \sigma_2 = \sigma_1 \sigma_{1,1} = \sigma_{2,1}$$

$$\sigma_2 \sigma_{1,1} = 0$$

$$\sigma_1 \sigma_1 = ?$$

Work to be done!

$$\sigma_1^2 = ?$$

$$\parallel$$

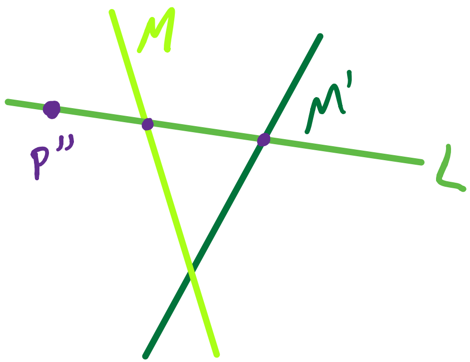
can be written as the linear comb. for unique $\alpha, \beta \in \mathbb{Z}$

$$\alpha \sigma_2 + \beta \sigma_{1,1}$$

Find α

$$\sigma_1^2 \sigma_2 = (\alpha \sigma_2 + \beta \sigma_{1,1}) \cdot \sigma_2$$

$$= \alpha \underset{\underset{\displaystyle \alpha\, \sigma_{2,2}}{=}}{\sigma_2^2} \left(+ \underset{\underset{\displaystyle 0}{=}}{\beta \sigma_{1,1} \sigma_2} \right)$$



$$\alpha = \# \left\{ \Lambda : \begin{array}{c} \Lambda \cap L \neq \emptyset \\ \Lambda \cap L' \neq \emptyset \\ p'' \in \Lambda \end{array} \right\} = 1 .$$

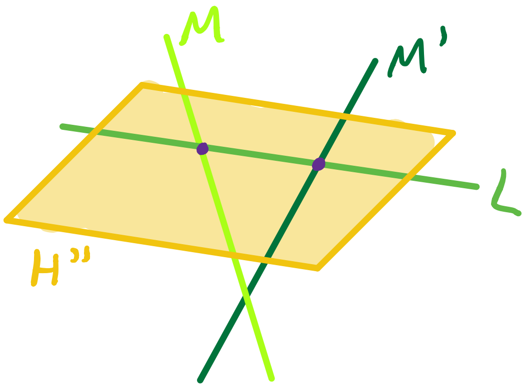
$$\Sigma_2(p'') \cap \Sigma_1(M) \cap \Sigma_1(M') = \{L\}$$

Find β

$$\sigma_1^2 \cdot \sigma_{1,1} = (\alpha\sigma_2 + \beta\sigma_{1,1})\cdot\sigma_{1,1}$$

$$= \alpha\sigma_2\cdot\sigma_{1,1} + \beta\sigma_{1,1}^2$$

$$= 0 + \beta\sigma_{2,2}$$



$\Sigma_{1,1}(H'')\cap\Sigma_2(M)\cap\Sigma_2(M) = \{L\}$

$$\beta = \#\left\{\Lambda : \begin{array}{c}\Lambda\cap L \neq \emptyset\\ \Lambda\cap L' \neq \emptyset\\ \Lambda \subset H''\end{array}\right\} = 1.$$

$$\boxed{\sigma_1^2 = \sigma_2 + \sigma_{1,1}}$$

Punchline:

The Schubert classes $\sigma_{i,j} = [\Sigma_{i,j}]$, $0 \le j \le i \le 2$ generate the Chow ring of $\mathbb{G}(1,3) = G(2,4)$ and satisfy the (multiplicative) relations

$$\sigma_2^2 = \sigma_{1,1}^2 = \sigma_1 \sigma_{2,1} = \sigma_{2,2}$$

$$\sigma_1 \sigma_2 = \sigma_1 \sigma_{1,1} = \sigma_{2,1}$$

$$\sigma_2 \sigma_{1,1} = 0$$

$$\sigma_1 \sigma_1 = \sigma_2 + \sigma_{1,1}$$

References

Coskun, Izzet. Lectures in Warsaw, Poland on homogeneous varieties, Lecture 1, Dec. 2013

Kleinman, Steven. "The transversality of the general translate."

Eisenbud, David Harris, Joe. "3264 & All That."