AG Working Group Oct 9

Tuesday, September 29, 2020 12:05 PM

Cohomology & Intersections :

$X = \mathbb{P}^n$ $[z_0:z_1:\dots:z_n]$

$z_i \neq 0$ $\mathbb{C}^n \subseteq \mathbb{P}^n$

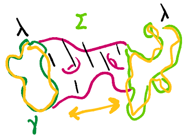
$z_i = 0$ $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ hyperplane

$Gr(k,n)$ k-dim'l subspace of $\mathbb{C}^n$

$Fl = GL(n,\mathbb{C})$ / upper triang. matr.

flag varieties in $\mathbb{C}^n$: $\{0\} \subseteq \mathbb{C} \subseteq \mathbb{C}^2 \subseteq \mathbb{C}^3 \subseteq \dots \subseteq \mathbb{C}^n$

Motivation: (i)



λ 1-form

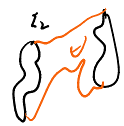
$$\int_\gamma \lambda = \int_{\gamma'} \lambda$$

Stokes thm if

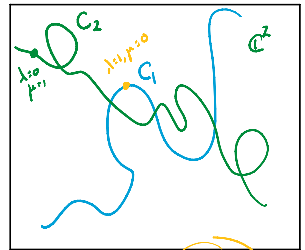
$$\int_\Sigma \omega = \int_\Sigma d\lambda = \int_{\partial\Sigma} \lambda$$

(ii) There's a well-defined thy of intersections for "cohomology classes":

$S \in H^{2i}(X)$ if S is a codim$_{\mathbb{C}}$-i subspace of X and there's the equiv. rel" $[\Sigma_1] = [\Sigma_2] = S$ if



Example: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $\mathbb{C}^2 \longrightarrow \mathbb{C}$



d degree

P_1, P_2 hom. poly. of deg d

$P_1, P_2 \in H^0(\mathbb{P}^2, \mathcal{O}(d))$

$C_1 = \{P_1 = 0\}$

$C_2 = \{P_2 = 0\}$

$\lambda P_1 + \mu P_2$, $\lambda, \mu \in \mathbb{C}$

$\mathbb{P}^1_{[\lambda:\mu]}$

$(z,w) \longmapsto f(z,w)$

$\mathbb{C}^2$

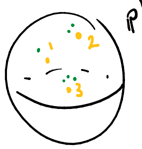
$\cup\!|$

$\{f=0\}, \{f=1\}$

$\{f = 3+i\}$

$f \downarrow$

Example: (1) $H^*(\mathbb{P}^1)$: here subvarieties are set of points $\{p_1, \dots, p_s\} \subseteq \mathbb{P}^1$



The total multiplicity of a set of points is determining

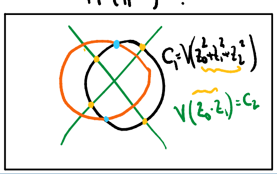
$d \in H^2(\mathbb{P}^1) \cong \mathbb{Z}$.

allow for multiplicity (divisor)

$\boxed{\text{the generator is a point which is the zero locus of a LINEAR POLY.}}$

In addition, $H^0(\mathbb{P}^1) \cong \mathbb{Z}$.

(2) $H^*(\mathbb{P}^2)$?



In fact, the degree of a curve determines its cohomology

$[C] \in H^2(\mathbb{P}^2) \cong \mathbb{Z}$

the degree.

the generator is $H = \{z_0 = 0\}$ a hyperplane

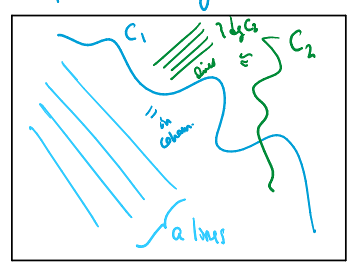
(cont'd) $H^0(\mathbb{P}^2) = \mathbb{Z}$, $H^2(\mathbb{P}^2) = \mathbb{Z}$, $H^4(\mathbb{P}^2) \cong \mathbb{Z}$

total space. curves and degree points degree

$\# |C_1 \cap C_2|$

Prop: (Bézout's thm) $[C_1], [C_2] \in H^2(\mathbb{P}^2) \cong \mathbb{Z}$, then $[C_1] \cdot [C_2] = \deg C_1 \cdot \deg C_2 \in H^4(\mathbb{P}^2)$.

Proof: $[C_1]$ of $\deg C_1 =: a$



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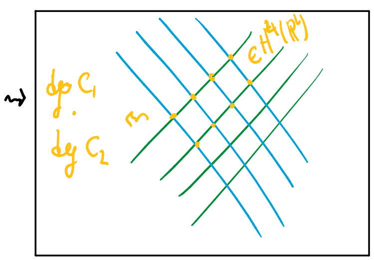

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(iii) $H^*(\mathbb{P}^n) = \dfrac{\mathbb{Z}\langle H\rangle}{(H^{n+1}=0)}$, H hyperplane $= \{l_i = 0\}$, l_i linear.

More generally $X = Gr(k,n)$ & flag varieties will rich intersection thy:

<u>Street math</u>: $n_q = 1+q+\dots+q^{n-1}$, then we have $n_q!$ is a polynomial.

even better $\binom{n_q}{k_q}$ is a polynomial.

<u>Thm</u>: (i) $H^{2i}(\mathcal{Fl}(\mathbb{C}^n))$ is free of rank the coeff. of $n_q!$ in front of q^i.

(ii) $H^{2i}(Gr(k,n))$ is free of rank the coeff. of $\binom{n_q}{k_q}$ in front of q^i.

<u>Ex</u>: $X = Gr(2,4)$, since $\dfrac{4_q!}{2_q!\,2_q!} = \binom{4_q}{2_q} = 1 + q + 2q^2 + q^3 + q^4$

so
$$\left\{\begin{array}{l} H^0(X) = \mathbb{Z} \\ H^2(X) = \mathbb{Z} \\ \boxed{H^4(X) = \mathbb{Z}^2} \\ H^6(X) = \mathbb{Z} \\ H^8(X) = \mathbb{Z} \end{array}\right.$$

2 dim'l, 1 dim'l

(2,3)

see Eugene's notes last week

$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

0-cell $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1-cell $\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2-cells $\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$ or $\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3-cell $\rightsquigarrow \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$

4-cell $\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$