On compact real or complex manifolds M of dimension n, submanifolds $X \subset M$ have fundamental classes in the top dimensional $H_i(X)$, with no sign ambiguity if the manifold is complex. By Poincare duality, you have a class $[X] \in H_i(M) \cong H^{n-i}(M)$. If X, Y intersect transversely, which means that at every point in their intersection the tangent spaces intersect transversely in T_pM, then the $[X] \cup [Y] = [X \cap Y]$. In Schubert calculus you want to do this where $M = Gr_k(\mathbb{C}^n)$, and the X's are Schubert varieties. Then problem is that they are singular and do not intersect transversely. The standard solution is Borel-Moore cohomology.

1 Borel-Moore homology

BM homology is the inverse limit

$$\bar{H}_i(X) = \lim_{\leftarrow K \subset X} H_i(X, X - K).$$

If you can imbed M in $\mathbb{R}^N$ for some N, then we have

$$\bar{H}_i(M) = H_{N-i}(\mathbb{R}^N, \mathbb{R}^N - M).$$

In particular, it's independent of the imbedding. If M is compact, then Borel-Moore homology agrees with normal homology.

Example 1. Compute the homology of the circle using the imbedding in $\mathbb{R}^2$. Contracting everything outside the circle to a point is like the sphere with two points glued, but it has no H_0 because of the relative homology/reduced homology.

Proposition 1 (From [1], Appendix B.3, Lemma 4)**.** *Let V be an algebraic subset of a nonsingular algebraic variety, and let k be the dimension of V. Then $H_i(V) = 0$ for $i > 2k$, and $H_{2k}(V)$ is a free abelian group with a generator for each k-dimensional irreducible component of V.*

Proof. You use the long exact sequence in relative cohomology

$$H^i(X, Y) \to H^i(X, Z) \to H^i(Y, Z) \to H^{i+1}(X, Y) \to \cdots$$

for any $Z \subset Y \subset X$, but for the subspaces

$$M - V \subset M - Z \subset M,$$

where Z is the singular set together with all components of lower dimension to reduce to the case of a manifold $V - Z$. $\square$

So each variety, which is has one component, has a unique generator in $\bar{H}_{2k}(V)$. Since there is a map

$$\bar{H}_i(V) \to \bar{H}_i(M),$$

which comes from the restriction map in relative cohomology, we obtain a class $[V] \in \bar{H}_{2k}(M)$.

Now we have to verify that intersections work as desired. Say that $X, Y \subset M$ intersect transversely if we can write

$$X \cap Y = \bigsqcup_i Z_i$$

where the codimension of Z_i is the sum of the codimensions of X, Y, and for every z in a Zariski open subset $U \subset Z_i$ we have that

$$T_z X \cap T_z Y = T_z Z_i$$

as vector subspaces of $T_z M$. The next proposition says that intersections work as desired for transverse intersections:

Proposition 2 (Fulton [1], equation (9))**.** *If $X, Y \subset M$ intersect transversely in the union of Z_i, then we have*

$$[X][Y] = \sum_i [Z_i]$$

Proof. The main point is that we have a cup product

$$\cup : \bar{H}_i(X) \times \bar{H}_j(Y) = H^{n-i}(M, M - X) \times H^{n-j}(M, M - Y) \to$$

$$H^{2n-i-j}(M, (M - X) \cup (M - Y)) = \bar{H}_{i+j-n}(X \cap Y)$$

which is compatible with the cup product on $\bar{H}_*(M) \cong H^*(M)$ by the induced maps. So, the only thing that remains to be shown is that the above map sends $([X], [Y])$ to $([X \cap Y])$, which is independent of the imbedding to M. $\square$

2 Schubert cells

We want to apply this to the case $M = \mathrm{Gr}_k(\mathbb{C}^n)$. First, we have

Proposition 3 ([1] Lemma 6 from B.6)**.** *If $\emptyset = X_0 \subset \cdots \subset X_d = X$ is a sequence of closed algebraic subsets of an algebraic variety X, such that $X_i \backslash X_{i-1}$ is a disjoint union of varieties $U_{i,j}$ each isomorphic to an affine space $\mathbb{C}^N$, then the classes $[\bar{U}_{i,j}]$ of the closures of these varieties give an additive basis for the Borel-Moore homology groups $\bar{H}_*(X)$ over $\mathbb{Z}$.*

Proof. Use the homology of $\mathbb{C}^N$ and the long exact sequence again. $\square$

The Schubert cells are such a decomposition for the Grassmannian. For each λ a partition that fits in a a k by $n - k$ box, let

$$U_\lambda = \{\text{rowspan}(A) : A \text{ is RREF and has a pivot at } k + i - \lambda_i \text{ in row } i\} \subset Gr_k(\mathbb{C}^n).$$

So for instance in $\mathrm{Gr}_3(\mathbb{C}^6)$, we have

$$U_{(2,1)} = \left\{ \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 \\ * & 0 & * & 0 & * & 1 \end{pmatrix} \right\},$$

noting that it's standard in Schubert calculus to have indices upside down. We let $V_\lambda = \bar{U}_\lambda$ be the closure in the topology of $Gr_k(\mathbb{C}^n)$, (which the next proposition shows is the same as the closure in the Zariski topology, if you're familiar with it).

Proposition 4 ([2] Proposition 3.2.3)**.** *For all partitions λ in a $k \times (n-k)$ box, we have*

1. The Schubert variety V_λ is an algebraic subvariety of $Gr_k(\mathbb{C}^n)$.
2. $U_\lambda \cong \mathbb{C}^{k(n-k)-|\lambda|}$
3. $V_\lambda = \bar{U}_\lambda = \bigsqcup_{\mu \supset \lambda} U_\mu$
4. $V_\lambda \supset V_\mu \Leftrightarrow \lambda \subset \mu$.

Proof. The second statement is done using coordinates on the Grassmannian. The next step is to write down equations of V_λ defined by the vanishing of certain minors of the matrix, see [2]. Let V'_λ be the variety defined by these equations. Check that statements 3,4 hold for V'_λ, which is a statement about sets. Then check that every point of

$$V'_\lambda = \bigsqcup_{\mu \supset \lambda} U_\mu$$

is a limit of points in U_λ using local coordinates, so that $V'_\lambda \subset V_\lambda = \bar{U}_\lambda$. But it's closed and contains U_λ, so we also have $V_\lambda \subset V'_\lambda$. □

Since each U_λ is isomorphic to affine space $\mathbb{C}^{k(n-k)-|\lambda|}$, Proposition 3 applies, and we find that

$$\sum_i \beta_{2i}(\mathrm{Gr}_k(\mathbb{C}^n))q^i = \sum_\lambda q^{k(n-k)-|\lambda|}$$

which is the q-binomial coefficient from Roger's talk.

So we have a basis $\bar{H}_*(Gr_k(\mathbb{C}^n)) \cong H^*(Gr_k(\mathbb{C}^n))$, and we'd like to know how to multiply and expand the cup product. The first issue is that they do not intersect transversely, so we can't apply Proposition 2 directly. But, we have the following:

Theorem 1 (Kleiman, see Coskun's notes from the webpage. Actually, the real theorem is much stronger)**.** *For generic matrices $g_i \in GL_n(\mathbb{C})$, we have that*

$$g_1 \cdot V_{\lambda^{(1)}}, \ldots, g_l \cdot V_{\lambda^{(l)}}$$

intersect transversely.

Proof. Won't begin to prove that, but the homework exercise shows that it is a testable statement in practice. □

This fixes the first problem of the transverse intersections. The next issue is that if we compute

$$g \cdot V_\lambda \cap V_\mu = \bigsqcup_i Z_i,$$

the Z_i are not themselves Schubert varieties, so we don't get an expansion

$$[V_\lambda] \cup [V_\mu] = \sum_\nu a^\nu_{\lambda,\mu} [V_\nu]$$

which is what we want. Instead what we need is to intersect triples to "extract" the coefficients $a^\nu_{\lambda,\mu}$. Let λ^* be the complement of λ in the box, then rotated 180 degrees, so $(2,1)^* = (3,2,1)$ in the 3×3 box for $Gr_3(\mathbb{C}^6)$.

Proposition 5. *We have that U_λ and $g \cdot U_{\lambda^*}$ intersect transversely in a single point, where g is the permutation matrix that puts all coordinates in reverse order.*

Proof. Just think about the example:

$$g \cdot U_{(3,2,1)} = \left\{ \begin{pmatrix} 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

which intersects $U_{(2,1)}$ transversely in one point. Checking transversality can be done in local coordinates. □

So then combining Theorem 1 with everything

$$[V_\lambda][V_\mu] = \sum_\nu a^\nu_{\lambda,\mu} [V_\nu],$$

and $a^\nu_{\lambda,\mu}$ is the number of points in

$$V_\lambda \cap g_1 \cdot V_\mu \cap g_2 \cdot V_{\nu^*},$$

so in particular, it's a nonnegative integer.

Theorem 2. *We have that $a^\nu_{\lambda,\mu}$ is the Littlewood-Richardson coefficient.*

Proof. Map $H_*(\mathrm{Gr}_k(\mathbb{C}^n))$ into symmetric functions modulo the ideal of all s_λ for λ not in the box, by sending $[V_\lambda]$ to s_λ. Then you find that this is a ring homomorphism (and so isomorphism) by checking that the Pieri rule holds on both sides. This is explained very well in [1]. □

References

[1] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996.

[2] L. Manivel and J. R. Swallow. Symmetric functions schubert polynomials and degeneracy loci. 2001.