

# Schubert calculus problem

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Fix  $n, k$  and let  $\Lambda = \Lambda^{(k)}$  denote the ring of symmetric functions in  $k$  variables  $x_1, \dots, x_k$  with coefficients in the integers. Recall that the Schur polynomial is defined as

$$s_\mu(x_1, \dots, x_k) = \sum_{T \rightarrow \mu} x_T$$

where  $T$  ranges over semistandard young tableaux filling the shape of  $\mu$  with the numbers  $\{1, \dots, k\}$ , and  $x_T$  is the product over all  $i \in T$  of  $x_i$ . Then the Schur polynomials  $s_\mu(x_1, \dots, x_k)$  for  $l(\mu) \leq k$  form a basis of  $\Lambda$ . Let  $I \subset \Lambda$  be the additive subgroup spanned by all Schur polynomials  $s_\mu$  with  $\mu_1 > n - k$ , which can be shown to be an ideal. Then the classes of the Schubert varieties  $[X_\mu]$  are a basis of  $H^*(Gr_k(\mathbb{C}^n))$ , and the map

$$H^*(Gr_k(\mathbb{C}^n)) \rightarrow \Lambda/I, \quad [X_\mu] \mapsto s_\mu$$

which sends the class of the Schubert variety  $[X_\mu]$  to  $s_\mu$  is an isomorphism of rings.

1. Check that in any number of variables, say  $k = 3$ , we have

$$s_{2,1}s_{2,1} = s_{4,2} + s_{4,1,1} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1} + s_{2,2,2} + s_{2,2,1,1}.$$

We'd like to calculate that coefficient of 2 using Schubert varieties. By the duality theorem on page 149 of Fulton, Young tableaux, the coefficient of 2 is the same as the number

$$[X_{2,1}][X_{2,1}][X_{2,1}] \in H^{2 \cdot 9}(Gr_3(\mathbb{C}^6)) \cong \mathbb{Z},$$

Since 9 is the dimension of the Grassmannian of 3 dimensional subspaces of  $\mathbb{C}^6$ , and the complementary partition of  $[3, 2, 1]$  in the  $3 \times 3$  box is also  $[2, 1]$ .

Straightforward, just use the formula to check

2. Let

$$U_{21} = \left\{ \left( \begin{array}{cccccc} a_{11} & 1 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 1 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} & 1 \end{array} \right) \right\}$$

be the Schubert cell, where the matrix is identified with its row span, and let

$$V_{21} = \left\{ \left( \begin{array}{cccccc} a_{11} & 1 & a_{13} & 0 & a_{15} & 0 \\ 0 & 0 & a_{23} & 1 & a_{25} & 0 \\ 0 & 0 & 0 & 0 & a_{35} & 1 \end{array} \right) \right\}.$$

Show that there is a matrix  $C \in GL_6(\mathbb{C})$  so that  $U_{21} \cdot C = V_{21}$ , where the product means the set of all  $A \cdot C$  for  $A \in U_{21}$ . In particular, we have that  $[\overline{U}_{21}] = [\overline{V}_{21}]$ . Check that

$$U = U_{21} \cap V_{21} = \left\{ \left( \begin{array}{cccccc} a_{11} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{35} & 1 \end{array} \right) \right\},$$

where the intersection is taken in the Grassmannian.

The matrix is the permutation matrix of  $(5, 6, 3, 4, 1, 2)$  in one-line notation. This doesn't take the representative matrix in  $U_{21}$  to the one in  $V_{21}$ , but it does if we multiply on the left by the permutation matrix of  $(3, 2, 1)$ , which doesn't affect the row span. Then both sets of matrices are in the chart

$$U_{21} = \left\{ \left( \begin{array}{cccccc} a_{11} & 1 & a_{13} & 0 & a_{15} & 0 \\ a_{21} & 0 & a_{23} & 1 & a_{25} & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} & 1 \end{array} \right) \right\}$$

and the intersection is precisely the matrices we are looking for.

3. Describe equations in the coefficients  $a_{ij}$  of a  $3 \times 6$  full rank matrix  $A$  that determine when the vector space spanned by  $A$  is in the Schubert

variety  $X_{21}$  with respect to the normal flag  $F_i = \langle e_1, \dots, e_i \rangle$ , using determinants of certain minors of  $A$  (see the proof of Proposition 3.2.3 in Manivel, or class notes from Friday for an answer).

It's the minors of the matrices from the book that determine the ranks of each subspace. In the case where  $A$  is  $3 \times 6$ , and  $\mu = [2, 1]$ , there are 8 of them:

$$[a_{15} a_{26} - a_{16} a_{25}, a_{15} a_{36} - a_{16} a_{35}, a_{25} a_{36} - a_{26} a_{35}, a_{14} a_{25} a_{36} - a_{14} a_{26} a_{35} - a_{15} a_{24} a_{36} + a_{15} a_{26} a_{34} + a_{16} a_{24} a_{35} - a_{16} a_{25} a_{34}, a_{13} a_{24} a_{35} - a_{13} a_{25} a_{34} - a_{14} a_{23} a_{35} + a_{14} a_{25} a_{33} + a_{15} a_{23} a_{34} - a_{15} a_{24} a_{33}, a_{13} a_{24} a_{36} - a_{13} a_{26} a_{34} - a_{14} a_{23} a_{36} + a_{14} a_{26} a_{33} + a_{16} a_{23} a_{34} - a_{16} a_{24} a_{33}, a_{13} a_{25} a_{36} - a_{13} a_{26} a_{35} - a_{15} a_{23} a_{36} + a_{15} a_{26} a_{33} + a_{16} a_{23} a_{35} - a_{16} a_{25} a_{33}, a_{14} a_{25} a_{36} - a_{14} a_{26} a_{35} - a_{15} a_{24} a_{36} + a_{15} a_{26} a_{34} + a_{16} a_{24} a_{35} - a_{16} a_{25} a_{34}]$$

4. Unfortunately, our open set  $U$  from the last part may not intersect this Schubert variety in exactly two points, since the points could be in the closure  $\bar{U}$ , or might not be transverse. The theorem of Kleiman says that we should be the right number of transverse intersection if we simply rotate the Schubert cells by multiplying by random elements in  $GL_6(\mathbb{C})$  on the right.

Take the following matrix, chosen basically at random:

$$C = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 2 & 2 \\ 2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Using the last part, find the points in the intersection  $X_{21} \cap (U \cdot C)$ . There should be two of them.

The matrix  $A \cdot C$ , where  $A$  is from the second part, is given by

$$\begin{bmatrix} 2a_{11} + 1 & 2a_{11} + 2 & a_{11} + 1 & 2a_{11} + 1 & a_{11} + 2 & 2a_{11} + 2 \\ 2 & a_{23} & 2 & 0 & 1 + 2a_{23} & 1 + 2a_{23} \\ 0 & 2 + 2a_{35} & a_{35} & 1 & 1 & 1 + a_{35} \end{bmatrix}$$

The equations from the second part on this matrix are

$$[-2a_{11}a_{23} - a_{11}, a_{11}a_{35} - a_{11} + 2a_{35}, 2a_{23}a_{35} + a_{35}, 4a_{11}a_{23}a_{35} - 2a_{11}a_{23} + 2a_{11}a_{35} + 2a_{23}a_{35} - a_{11} + a_{35}, 4a_{11}a_{23}a_{35} - 2a_{11}a_{23} + 2a_{11}a_{35} + 2a_{23}a_{35} - 3a_{11} - 2a_{23} + a_{35} + 1, 4a_{11}a_{23}a_{35} - 2a_{11}a_{23} - 2a_{11}a_{35} + 2a_{23}a_{35} - a_{11} - 2a_{23} - a_{35} + 1, -2a_{11}a_{35} + 2a_{23}a_{35} + 2a_{11} - 3a_{35}, 4a_{11}a_{23}a_{35} - 2a_{11}a_{23} + 2a_{11}a_{35} + 2a_{23}a_{35} - a_{11} + a_{35}].$$

Solving these equations gives exactly two answers,

$$\{a_{11} = 0, a_{23} = 1/2, a_{35} = 0\}, \{a_{11} = 1, a_{23} = -1/2, a_{35} = 1/3\}$$

So the two matrices are

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1 \end{bmatrix}$$

5. Show that this intersection is transverse at both points. Taking the two matrices above, just check the rank of the Jacobian matrix at each point, using the equations in the variables  $a_{12}, a_{23}, a_{35}$ .