Meeting 2/11

Thursday, February 11, 2021 10:47 AM

<u>Recap</u>: Spec A = {all prime ideals in A}

Zariski topology:
<u>closed subsets</u> $V(I)$ → zero locus of I (in $\mathbb{C}^n$)
→ prime ideals in Spec A <u>containing</u> I

Open subsets = complements to closed

<u>Lemma</u> A point $\{p\} \in$ Spec A is closed iff p is a maximal ideal.
(p ← prime ideal)

<u>Proof</u>: (a) p maximal $\Rightarrow V(p) = \{$prime ideals in A containing $p\}$ $= \{p\}$ ← because p is maximal

(b) p prime ideal $\Rightarrow \exists$ maximal ideal $m \supset p$
If $\{p\} = V(I)$ for some I, then
$$m \supset p \supset I$$
so m contains I, and $m \in V(I) \Rightarrow m = p$. ∎

Distinguished open subsets: <u>$f \in A$</u>

$D(f) = \{f \neq 0\} \subset$ Spec A
$= \{$prime ideals <u>not</u> containing $f\}$

$\setminus \left\{ \begin{array}{l} \text{prime ideals} \\ \underline{\text{not}} \text{ containing } f \end{array} \right\}$

$D(f)$ = complement to $\{f=0\} = V(f)$

open ↑ $D(f)$; closed ↑ $V(f)$

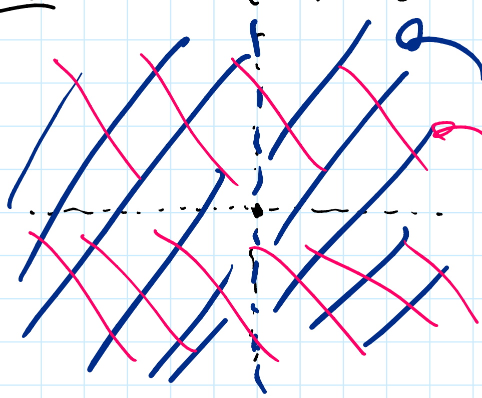
<u>Lemma</u>: (a) $D(f)$ is Zariski open in Spec A

(b) Any Zariski open is union of $D(f_i)$

(c) $D(f) \cap D(g) = D(fg)$.

<u>Rmk</u> This shows that $D(f)$ form a <u>base</u> of Zariski topology.

<u>Ex</u> $\mathbb{C}^2 \setminus \{(0,0)\}$ = open in $\mathbb{C}^2$



$D(x) = \{x \neq 0\}$

$D(y) = \{y \neq 0\}$

$D(x) \cup D(y) = \mathbb{C}^2 \setminus \{(0,0)\}$

$D(x) \cap D(y) = \{x \neq 0 \text{ AND } y \neq 0\}$

$= \{xy \neq 0\} \overset{\shortparallel}{D}(xy)$

We can cover $\mathbb{C}^2 \setminus \{(0,0)\}$ by two distinguished open subsets, intersection $= D(xy)$

<u>Proof</u>: (a) $V(f) = \left\{ \begin{array}{l} \text{prime ideals} \\ \text{containing } (f) \end{array} \right\} = \left\{ \begin{array}{l} \text{prime ideals} \\ \text{containing } f \end{array} \right\}$ ← closed

$D(f) = \left\{ \begin{array}{l} \text{prime ideals} \\ \underline{\text{not}} \text{ containing } f \end{array} \right\} = \operatorname{Spec} A \setminus V(f)$

↑ open

$\{$ not containing ... $\}$ ← open

(b) I ideal generated by (f_i)

$$V(I) = \{\text{prime ideals containing } \underline{\text{all}}\ f_i\} = \bigcap_i V(f_i)$$

$\{$all $f_i = 0\}$

Open $\operatorname{Spec} A \setminus V(I) = $ complement to $\bigcap_i V(f_i)$

$$= \bigcup_i D(f_i)$$

$\{$at least one $f_i \neq 0\}$ $= \bigcup_i \{f_i \neq 0\}$

(c) $D(fg) = \{\text{prime ideals not containing } fg\}$

$$p \text{ prime } \ni fg \iff p \ni f \text{ or } p \ni g$$

$$V(fg) = V(f) \cup V(g)$$

$$D(fg) = D(f) \cap D(g).$$

How to think of functions on $D(f)$?

Localization of A in f ← ring $\quad$ $f \in A$ fixed

$$A_f = \left\{ \frac{g}{f^k} : g \in A,\ k \geq 0 \right\} / \sim$$

allowed to divide only by powers of f

$$\frac{g_1}{f^k} \sim \frac{g_2}{f^\ell} \text{ if } f^\ell g_1 = g_2 f^k \text{ in } A.$$

Ex: $A = \mathbb{C}[x]$ $\quad f = x$

$$A_x = \left\{ \frac{g(x)}{x^k} \right\} = \mathbb{C}[x, x^{-1}] \quad \text{Laurent polynomials.}$$

$$\frac{x^2 + 3x + 5}{x} = x + 3 + 5x^{-1}$$

$$\frac{x^2+3x+5}{x} = x+3+5x^{-1}$$

Laurent polynomials are well-defined functions on $\{x \neq 0\} = \mathbb{C}^* \subset \mathbb{C}$

$$\{x\neq 0\} = D(x)$$

$\mathbb{C}[x,x^{-1}]$ = ring of algebraic function on $\{x \neq 0\}$.

A_f = ring of algebraic function on $D(f) = \{f \neq 0\}$

In other words, $\boxed{D(f) = \operatorname{Spec} A_f}$

<u>Lemma</u>. A_f is a ring

- There is a ring homomorphism $A \to A_f$
- This corresponds to the map $D(f) = \operatorname{Spec} A_f \to \operatorname{Spec} A$

<u>Proof</u>:

- $$\frac{g_1}{f^k} + \frac{g_2}{f^l} = \frac{g_1 \cdot f^l + g_2 \cdot f^k}{f^{k+l}}$$
- $$\left(\frac{g_1}{f^k}\right)\cdot\left(\frac{g_2}{f^l}\right) = \frac{g_1 g_2}{f^{k+l}}$$

operations with fractions preserve condition on denominator

- $g \longrightarrow \frac{g}{1}$ homomorphism.

<u>Ex</u>: $A = \frac{\mathbb{C}[x,y]}{(xy)}$ $\quad f = x$

Algebraically: $A_f = \left\{ \dfrac{g(x,y)}{x^k} \right\}$ mod relation

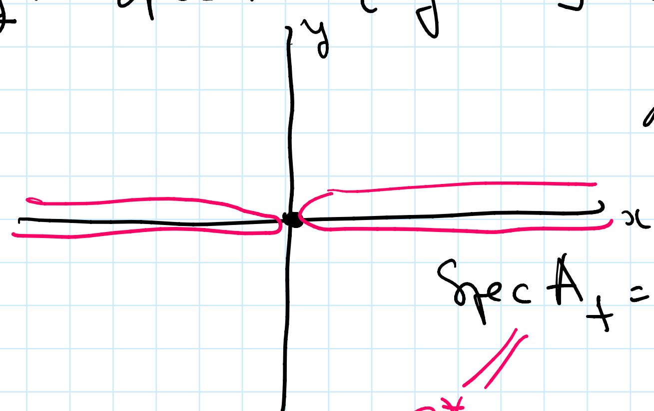
$$xy = 0 \;\Downarrow\; x^{-1}xy = 0 \;\Downarrow\; y = 0$$

$A \to A_f$ is not injective ($y \in \ker$)

there could be more relations in the localization than in the original ring

$$A_f = \left\{ \frac{g(x)}{x^k} \right\} = \mathbb{C}[x, x^{-1}] \qquad \operatorname{Spec} A_f = \mathbb{C}^*.$$

Geometrically: $\operatorname{Spec} A = \{xy = 0\}$ = union of two axis



$\operatorname{Spec} A_f$ = subset $\{f \neq 0\} \subset \operatorname{Spec} A$

$\{x \neq 0\} = D(x)$

$\mathbb{C}^*$

= (union of two axis) − (y-axis)

= (x-axis) − $\{0\}$ = $\mathbb{C}^*$.

Another perspective: $A_f = \dfrac{A[z]}{(zf = 1)} \longleftrightarrow z = \dfrac{1}{f}$

$\dfrac{f}{g^k} \leftrightarrow f z^k$

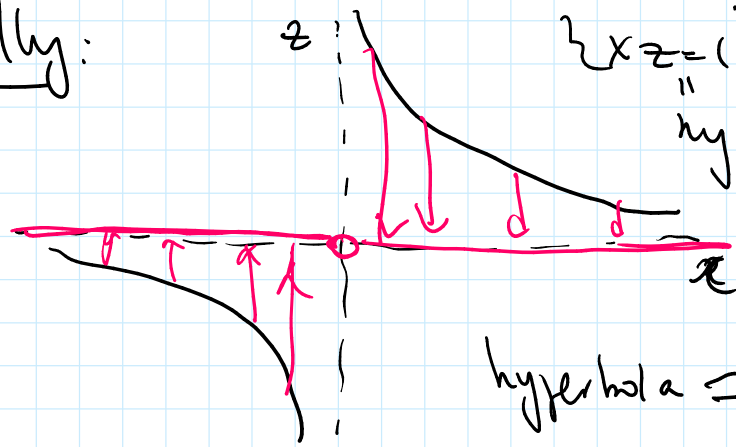
Cor: $D(f) \simeq$ closed subset in $\operatorname{Spec} A[z]$

Ex $A = \mathbb{C}[x]$ $f = x$

$$A_f = \mathbb{C}[x, x^{-1}] = \mathbb{C}[x, z]$$

$$A_f = \mathbb{C}[x, x^{-1}] \simeq \frac{\mathbb{C}[x,z]}{(xz-1=0)}$$

Geometrically:



$\{xz=1\}$ = hyperbola

hyperbola $\simeq \{x \neq 0\}$

for any x, there's unique $z = 1/x$.

$D(f)$ = open in Spec A $\simeq$ closed in Spec $A[z]$