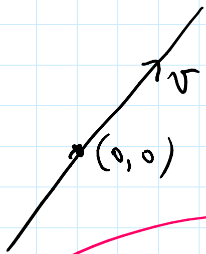
Meeting 2/18

Thursday, February 18, 2021 10:58 AM

# Projective space

① $\mathbb{K}$ = any field $\qquad \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{K}}$ = projective line

$\mathbb{P}^1$ = set of all lines in $\mathbb{K}^2$ through the origin (= 1d vector subspaces of $\mathbb{K}^2$)



$\vec{v} = (x, y)$ = non zero vector

$(0,0)$

- $x, y$ do not vanish simultaneously
- $(x,y) \sim (\lambda x, \lambda y) \quad \lambda \neq 0$

two vectors define same line

$\mathbb{P}^1$ = set of equivalence classes for this equivalence relation.

$$\mathbb{P}^1 = \frac{\mathbb{K}^2 \setminus \{(0,0)\}}{\mathbb{K}^*}$$

$\mathbb{K}^*$ acts on $\mathbb{K}^2$ ($\lambda$)

$\mathbb{K}^*$ acts freely on $\mathbb{K}^2 \setminus \{(0,0)\}$

$[x : y]$ homogeneous coordinates on $\mathbb{P}^1$.

How to visualize $\mathbb{P}^1$?

(a) For $y \neq 0 \quad [x:y] \sim [\frac{x}{y} : 1] \simeq \mathbb{K}$ line

$y = 0 \Rightarrow x \neq 0 \quad [x : 0] \sim [1 : 0] \simeq \{\infty\}$

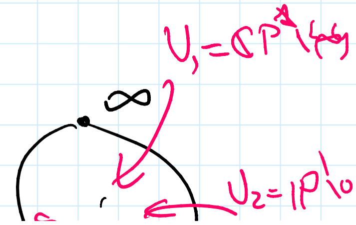
$$\mathbb{P}^1 = \mathbb{K} \cup \{\infty\}$$

(ex. $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\} = S^1$

$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \simeq S^2$ )

(b) Two open charts:

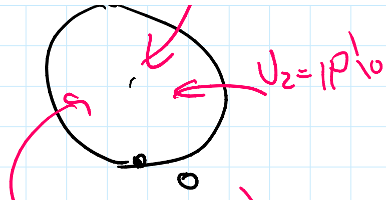
$U_1 : y \neq 0 \qquad [x : 1] \simeq \mathbb{K}$




$U_2 : x \neq 0 \qquad [1 : y] \simeq \mathbb{K}$

$U_1 : y \neq 0 \qquad [x:1] \simeq \mathbb{K}$

$U_2 : \underbrace{x \neq 0} \qquad [1:y] \simeq \mathbb{K}$




$U_1 \cap U_2 = \mathbb{P}^1 \setminus \{0, \infty\}$

$\parallel$

$\mathbb{K}^*$

$\cap$

$\mathbb{K} \setminus \{0\}$

$U_1 \cap U_2 = \mathbb{K}^* = \{x \neq 0 \text{ AND } y \neq 0\}$

$U_1 \cap U_2$ is Zariski open in both $U_1$ and $U_2$

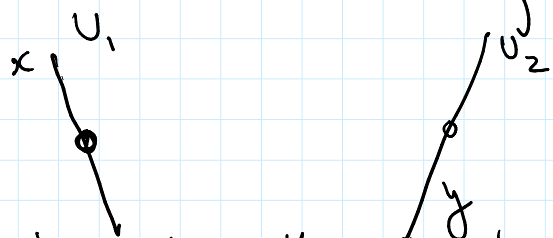
$$\begin{tikzcd} & \mathbb{P}^1 \arrow[dr, leftrightarrow] & \\ {[x:1]}\ U_1 \arrow[ur, hook] & & U_2\ {[1:y]} \\ & U_1 \cap U_2 \arrow[ul, "x \neq 0"] \arrow[ur, hook, "y \neq 0"'] & \end{tikzcd}$$

Zar. open condition

On $U_1 \cap U_2$ we have coordinate $x$ (from $U_1$) and $y$ from $U_2$)

They are related as following: $[x:1] \sim [1:y]$ $\quad y = 1/x$

This determines the "gluing" of $U_1$ and $U_2$.




We glue the subset $\underset{\text{w. coord } x}{U_1 \setminus \{0\}}$ with $\underset{\text{w. coord } y}{U_2 \setminus \{0\}}$ by the function $y = 1/x$

**Thm**: Any global (algebraic) function on $\mathbb{P}^1$ is a constant.

**Def** A function $f : \mathbb{P}^1 \to \mathbb{K}$ is algebraic if its restriction to $U_1$ and $U_2$ are polynomials.

**Proof** $f : \mathbb{P}^1 \to \mathbb{K}$

$\varphi_1(x)$ = restriction of $f$ to $U_1$

$\varphi_2(y)$ = restriction of $f$ to $U_2$

$$\begin{tikzcd} & \mathbb{P}^1\ f & \\ U_1\ \varphi_1(x) \arrow[ur, hook] & & U_2\ \varphi_2(y) \arrow[ul] \\ & U_1 \cap U_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\varphi_1(x) = a_n x^n + a_{n-1} x^{n-1} + \dots$

$$\varphi_1(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$$

$U_1 \cap U_2$

$\varphi_1, \varphi_2$ agree on $U_1 \cap U_2$

On $U_1 \cap U_2$ we have $y = 1/x$, so

$$\varphi_1 = \frac{a_n}{y^n} + \frac{a_{n-1}}{y^{n-1}} + \dots + a_0$$

this extends to a polynomial on $U_2$

iff $a_n = a_{n-1} = \dots = a_1 = 0$.

<u>Rmk</u> A function (structure...) is determined by its behaviour in charts $U_1$ and $U_2$ provided that the restrictions to $U_1$ an $U_2$ agree on $U_1 \cap U_2$.

[$\Rightarrow$ notion of a sheaf]

<u>Rmk</u>: <u>Liouville's Thm in Complex Analysis</u>: a bounded holomorphic function on $\mathbb{C}$ is a constant.

Holom. function on $\mathbb{CP}^1$ $\rightsquigarrow$ bounded holom. fn. on $\mathbb{C}$.

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(2) $\mathbb{P}^n$ = space of lines in $\mathbb{k}^{n+1}$

$[x_0 : x_1 : \dots : x_n] \sim [\lambda x_0 : \lambda x_1 : \dots : \lambda x_n]$ $\lambda \neq 0$

homogeneous coordinates.

<u>Open charts</u>: $U_i = \{x_i \neq 0\}$ $i = 0, \dots n$

$$[x_0 : \dots : x_i : \dots : x_n] \sim \left[\frac{x_0}{x_i} : \dots : 1 : \dots \frac{x_n}{x_i}\right].$$

$U_i = \mathbb{k}^n$

$\mathbb{k}^* \times \mathbb{k}^{n-1}$ is $U_i \cap U_j$

$U_i \cap U_j \xrightarrow{a_j \neq 0} U_i = [a_0 : \dots : \underset{i}{1} : \dots : a_j : \dots : a_n] \simeq \mathbb{k}^n$

$U_i \cap U_j \xrightarrow{b_i \neq 0} U_j = [b_0 : \dots : b_i : \dots : 1 : \dots : b_n] \simeq \mathbb{k}^n$

Zariski open in both $U_i, U_j$

We can describe the change of coordinates / gluing:

We can describe the change of coordinates (gluing):

On $U_i \cap U_j$ we have $b_k = \frac{a_k}{a_j}$ $k \neq i, j$ $\quad b_i = \frac{1}{a_j}$

Conversely, $a_k = \frac{b_k}{b_i}$, $a_j = \frac{1}{b_i}$ $\quad k \neq i, j$

All these are algebraic functions on $\{a_j \neq 0\}$

and $\{b_i \neq 0\}$.

<u>Ex</u>

$$
\begin{tikzcd}
 & \mathbb{P}^2 & \\
U_0 \; [1:a_1:a_2] \arrow[ur, hook] & U_1 \; [b_0:1:b_2] \arrow[u, hook] & U_2 \; [c_0:c_1:1] \arrow[ul, hook] \\
U_0 \cap U_1 \arrow[u, hook, "a_1 \neq 0"] \arrow[ur, hook, "b_0 \neq 0"] & U_0 \cap U_2 \arrow[ul, hook, "a_2 \neq 0"] \arrow[ur, hook, "c_0 \neq 0"] & U_1 \cap U_2 \arrow[ul, hook, "b_2 \neq 0"] \arrow[u, hook, "c_1 \neq 0"] \\
 & U_0 \cap U_2 \cap U_3 \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

all $\mathbb{K}^2$

all $\mathbb{K}^* \times \mathbb{K}$

— $\mathbb{K}^* \times \mathbb{K}^*$ (all homogeneous coords $\neq 0$).

$$\underbrace{b_0 = \frac{1}{a_1} \quad b_2 = \frac{a_2}{a_1}}_{U_0 \dashrightarrow U_1}$$

$$c_0 = \frac{1}{a_2} = \frac{b_0}{b_2} \qquad c_1 = \frac{a_1}{a_2} = \frac{1}{b_2}$$

<u>Thm</u>: Any global function on $\mathbb{P}^n$ is a constant

<u>Proof</u>: <u>Same</u>. (exercise).

<u>Q</u>: How to do algebraic geometry in $\mathbb{P}^n$?

<u>A1</u>: Work locally in charts

<u>A2</u>: Use homogeneous equations & ideals.

<u>Ex</u>: $\{ x_0^2 + x_1^2 - x_2^2 = 0 \} \subset \mathbb{P}^2$

This is a well defined subset of $\mathbb{P}^2$!

$[x_0 : x_1 : x_2] \sim [\lambda x_0 : \lambda x_1 : \lambda x_2]$ $\quad \lambda^2 x_0^2 + \lambda^2 x_1^2 - \lambda^2 x_2^2 = 0$

$$[x_0 : x_1 : x_2] \sim [\lambda x_0 : \lambda x_1 : \lambda x_2] \qquad \lambda^2 x_0^2 + \lambda^2 x_1^2 - \lambda^2 x_2^2 = 0$$

$$x_0^2 + x_1^2 - x_2^2 = 0 \iff \lambda^2 (x_0^2 + x_1^2 - x_2^2) = 0$$

In general, if $f(x_0, \dots x_n)$ is a homogeneous polynomial then $\{f(x_0, \dots x_n) = 0\}$ is a well defined subset of $\mathbb{P}^n$. Moreover, we can consider

$$\mathbb{P}^n \supset \{f_1(x_0, \dots x_n) = 0, f_2(x_0, \dots x_n) = 0, \dots f_k(x_0, \dots x_n) = 0\}$$

$f_1 \dots f_k$ = homogeneous polynomials (possibly of different degrees).

$\mathbb{R}P^2$ We can look in charts:

$U_0 : [1 : a_1 : a_2] \dots \quad \{1^2 + a_1^2 - a_2^2 = 0\}$

$U_1 : [b_0 : 1 : b_2] \quad \{b_0^2 + 1^2 - b_2^2 = 0\}$

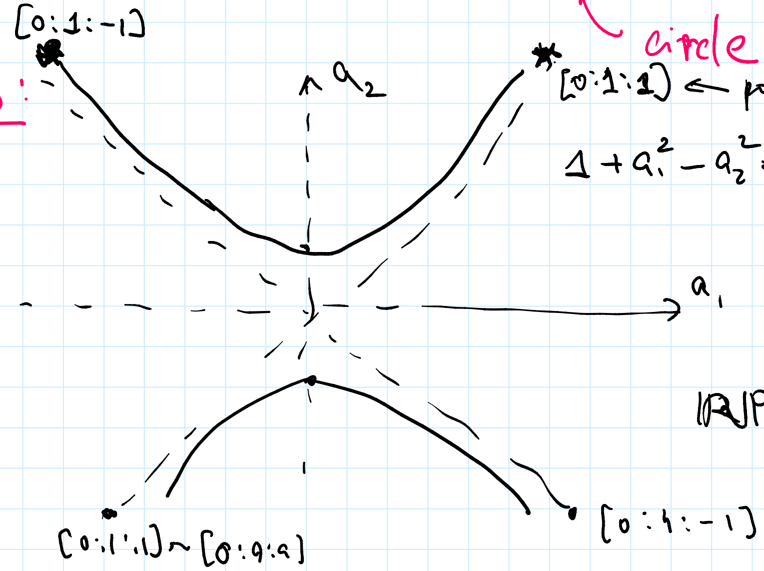
$U_2 : [c_0 : c_1 : 1] \quad \{c_0^2 + c_1^2 - 1^2 = 0\}$

3 different algebraic varieties in $U_i$ which agree on the intersection.

hyperbolas

circle

Zoom in $U_0$:




$\mathbb{R}P^2 = \mathbb{R}^2 \cup \{\text{infinite line}\}$

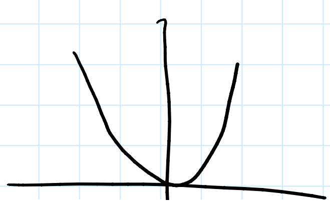
Note: Complement to $U_i = \mathbb{R}P^{n-1}$

Comp. to $U_0 = [0 : a_1 : a_2]/\sim = \mathbb{R}P^1$

But: $x_0^2 + x_1^2 - x_2^2$ is not a function

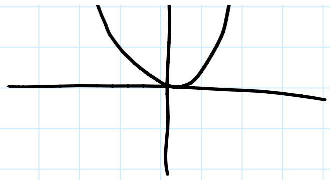
What is it? Next time :)

Exercise



$a_2 = a_1^2$ parabola in chart $U_0$

(a) Find a homogeneous equation



(a) Find a homogeneous equation in $[x_0 : x_1 : x_2]$ defining parabola

(b) Describe the points at $\infty$.