Meeting 2/25

Thursday, February 25, 2021 11:07 AM

$x_1^2 = x_0 x_2$

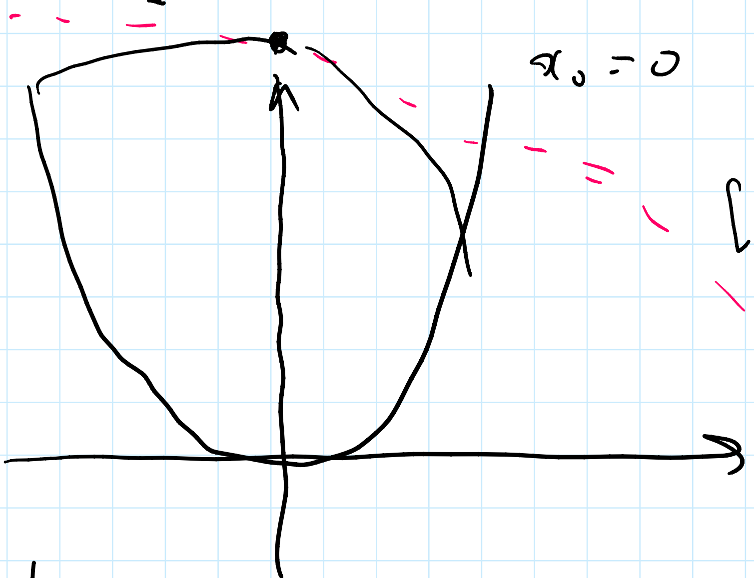
Chart U_1

$x_0 = 1 \quad x_1^2 = x_2$

$x_0 = 0 \quad x_1^2 = 0$

one pt

$[0:0:1]$



U_1: $x_1 = 1 \qquad x_0 x_2 = 1 \Rightarrow$ hyperbola

U_2: $x_2 = 1 \qquad x_1^2 = x_0 \Rightarrow$ parabola.

Line bundles

X = some space ($\mathbb{P}^1$, $\mathbb{P}^n$)

Def A line bundle on X is the following data: (a) The "total space" V

(b) The projection $\pi : V \longrightarrow X$

such that for all $x \in X$ the preimage $\pi^{-1}(x)$ is a 1-dimensional vector space

(c) [illegible]

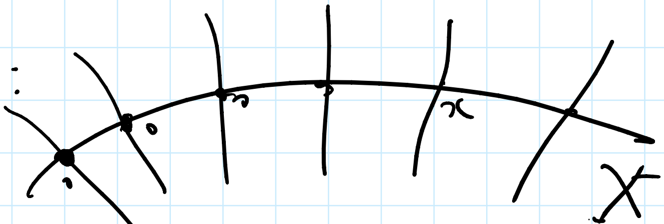
vector space

(c) Locally trivial condition:

for each $x \in X$ there is an open subset $U \ni x$ such that

$\pi^{-1}(U) \simeq U \times \mathbb{K}$

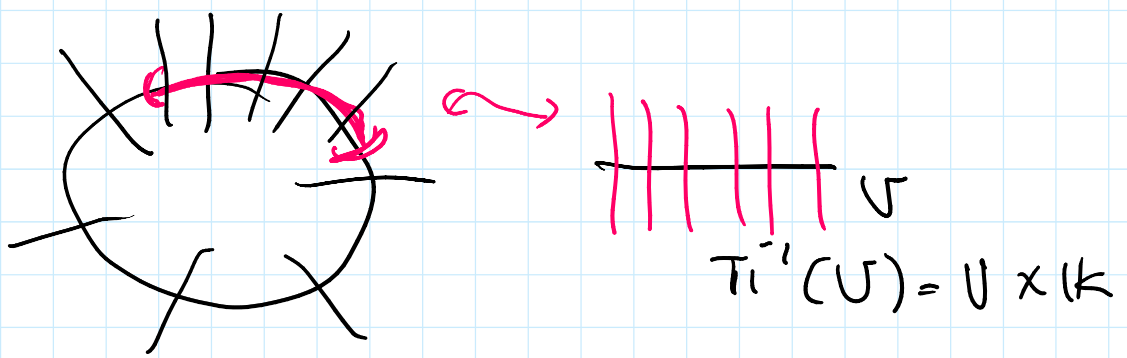
$\mathbb{K}$: 1d vector space



line $\pi^{-1}(x)$ at each base point of x, V = union of all these lines

<u>Ex</u> Trivial line bundle $V = X \times \mathbb{K}$

π = projection to X.



<u>Note</u>: Can use both "usual" and Zariski topology for definition,

<u>Ex</u> Tautological bundle on $\mathbb{P}^1$

$$V = \{(\ell, p) \mid p \in \ell\} \subset \mathbb{P}^1 \times \mathbb{K}^2$$

ℓ: line in $\mathbb{K}^2$ $\quad$ p: point in $\mathbb{K}^2$

line in $\mathbb{K}^2$ point in $\mathbb{K}^2$

$$\Big\downarrow \pi$$

$\pi(\ell, p) = \ell$ (forget p)

$\mathbb{P}^1$ Given $\ell \in \mathbb{P}^1$,

$$\pi^{-1}(\ell) = \left\{ \begin{array}{c} \text{all points } p \\ \text{on } \ell \end{array} \right\} \simeq \mathbb{K}$$

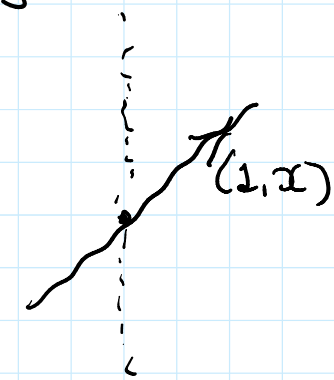
1 d vector space

Why (c) holds?

Recall: $[x_0 : x_1]$ homogeneous coordinates on $\mathbb{P}^1$

$U_0 = \{x_0 \neq 0\}$

$[x_0 : x_1] = [1 : x]$ $\quad x = \frac{x_1}{x_0}$



All points on the line

$\ell = \operatorname{Span}(1, x)$ have the form $(t_0, t_0 x)$

t_0 any

$$(\ell, p) \longleftrightarrow ([1 : x], t_0)$$

$$\underset{U_0 \times \mathbb{K}}{\cap\!\!\!\!\!\;\text{\scriptsize{|}}}$$

t_0 is the coordinate in the fiber

$U_1 = \{x_1 \neq 0\}$

$[y : 1]$, $y = \frac{x_0}{x_1}$

$U_1 \times \mathbb{K}$

$([y : 1], t_1)$

$(\ell, p = (t_1 y, t_1))$

$(\ell, p = (t, y, t_1))$

How do glue? $p = (t_0, t_1 x)$

On $U_0 \cap U_1$ $= (t_1 y, t_1)$

$$\boxed{t_1 = x t_0 = \frac{x_1}{x_0} t_0}$$

In general: (1) Choose open charts U_i where π is trivial

t_i = coordinate in the fiber over U_i

(2) Transition functions $t_j = \varphi_{ij}(x) t_i$ where φ_{ij} are some invertible functions on $U_i \cap U_j$

(3) "Cocycle condition" $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$ $\quad \varphi_{ij}\varphi_{ji} = 1 \quad \varphi_{ji} = \varphi_{ij}^{-1}$ $(*)$

Fact: A line bundle is completely determined by the set of φ_{ij} satisfying $(*)$.

Ex Tautological bundle on $\mathbb{P}^2$

$V = \{(\ell, b) \mid b \in \ell\} \subset \mathbb{P}^2 \times \mathbb{k}^3$

$$V = \{(\ell, p) \mid p \in \ell\} \subset \mathbb{P}^2 \times \mathbb{K}^3$$

$$\ell = [x_0 : x_1 : x_2]$$

$$U_0 = \left[1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}\right] \qquad U_1 = \left[\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right]$$

$$p = \left(t_0, \frac{x_1}{x_0} t_0, \frac{x_2}{x_0} t_0\right) = \left(t_1 \frac{x_0}{x_1}, t_1, t_1 \frac{x_2}{x_1}\right)$$

$$t_1 = \frac{x_1}{x_0} t_0 \qquad \varphi_{01} = \frac{x_1}{x_0}$$

Same on $\mathbb{P}^n$: charts $U_i = \{x_i \neq 0\}$

$$\boxed{t_j = \frac{x_j}{x_i} t_i}$$

Serre's twisting bundle

Ex Line bundle $\boxed{\mathcal{O}(k) \text{ on } \mathbb{P}^n}$

Define by transition functions

$$t_j = \left(\frac{x_i}{x_j}\right)^k t_i \qquad k \in \mathbb{Z} \text{ fixed.}$$

well defined and invertible on $U_i \cap U_j$

Exercise (a) Check cocycle condition

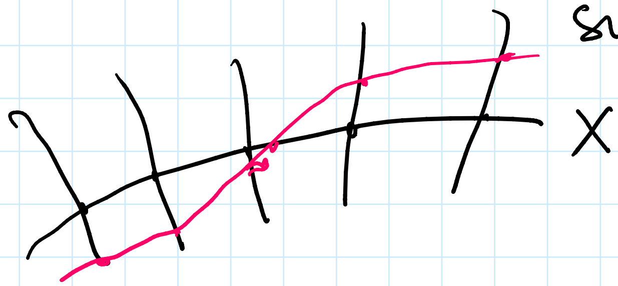
(b) Tautological $= \mathcal{O}(-1)$

(c) For $k \geq 0$, the fiber of $\mathcal{O}(k)$ over $\ell \in \mathbb{P}^n$ = degree k homogeneous polynomials?

over $\ell \in \mathbb{P}^n = \{$ degree k homogeneous polynomials on $\ell \}$
$\cong (\ell^*)^{\otimes k}$

(d) For $k \le 0$, the fiber $\ell^{\otimes k}$.

Def A *section* of V is a function $s : X \to V$ such that $\pi(s(x)) = x$ for all x



$s \longleftrightarrow$ choose one point in each fiber of π

Ex $\mathbb{P}^1$, $\mathcal{O}(k)$: a section over U_0 is an algebraic function ($=$ polynomial)

$t_0 = s_0(x)$ $\qquad$ $s : U_0 \longrightarrow U_0 \times \mathbb{K}$
$\qquad\qquad\qquad x \longmapsto (x, s_0(x))$

A section over U_1 : $t_1 = s_1(y)$

When do they glue to a global section of $\mathcal{O}(k)$ on $\mathbb{P}^1$?

Gluing: $t_1 = \left(\dfrac{x_0}{x_1}\right)^k t_0$ $\qquad$ $x = \dfrac{x_1}{x_0}$

$s_1(y) = x^{-k} s_0(x)$

$s_1(y) = x \quad s_0(x)$

$s_1(\frac{1}{x})$ When this is possible?

$k<0$ $\Rightarrow$ never happens! RHS polynomial in x with all positive degrees.

$k>0$ This is possible if $\boxed{\deg s_0(x) \leq k}$

Conclusion: • For $k<0$ the line bundle $\mathcal{O}(k)$ have no global sections (only $s=0$)

• For $k>0$ $\mathcal{O}(k)$ has $(k+1)$ dim space of global sections on $\mathbb{P}^1$

$s(x_0, x_1)$ = homogeneous polynomial in x_0, x_1 of degree k.

$$s(x_0, x_1) = x_0^k \underbrace{s(1, \tfrac{x_1}{x_0})}_{s_0(x)} = x_1^k \underbrace{s(\tfrac{x_0}{x_1}, 1)}_{s_1(y)}$$

$$s_1(y) = \frac{x_0^k}{x_1^k} s_0(x)$$

$$\left\{ \begin{array}{c} \text{global sections} \\ \text{of } \mathcal{O}(k) \text{ on } \mathbb{P}^1 \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{homogeneous} \\ \text{polynomials} \\ \text{of degree } k \text{ in } x_0, x_1 \end{array} \right\}$$

For $k=0$ In this case $t_{..} = t_{..}$ $\mathcal{O}(0) = \mathcal{O}$ is

Ex $k=0$ In this case $t_0=t_1$, $\mathcal{O}(0)=\mathcal{O}$ is the trivial line bundle

Sections of $\mathcal{O}$ = functions on $\mathbb{P}^1$

Last time: any global function on $\mathbb{P}^1$ is a constant

Today: global function on $\mathbb{P}^1$ = homogeneous polynomial of degree 0 = constant.

Morally, if we work in $\mathbb{P}^n$, we always need to consider line bundles and their sections, not only functions (≈ locally look like functions).

$s: X \to V$ section $\quad \{s=0\} \subset X$ closed subset of X

Ex $V = \mathcal{O}(k)$ $\quad \{s(x_0 : \ldots : x_n) = 0\} \subset \mathbb{P}^n$

s homogeneous polynomial of degree k

Rmk (a) Any line bundle on any space has zero section $S \equiv 0$.

(b) Sections of V form a vector space:

s_1, s_2 = sections $\rightsquigarrow$ $s_1 + s_2$ = section

$s \cdot f(x)$ = section $f(x)$ = any function.

$\Gamma(X, V)$ = space of global sections

$\shortparallel$

$H^0(X, V)$