Meeting 1/14

Saturday, January 9, 2021 3:13 PM

Alexander Grothendieck (1928-2014)

Schemes (Vakil 3.2)

A = commutative ring

Spec A = $\left\{ \begin{array}{c} \text{all prime} \\ \text{ideals in } A \end{array} \right\}$ points of Spec A

affine scheme (so far just a set)

$\cup$

$\left\{ \begin{array}{c} \text{all maximal} \\ \text{ideals in } A \end{array} \right\}$

closed points

Recall

- $I \subset A$ is <u>prime</u> if $fg \in I \Rightarrow$ f in I or g in I

$I \neq A$

- $I \subset A$ is <u>maximal</u> if $J \supset I$ ideal then $J = A$ or $J = I$

Exercise (a) $I \subset A$ is prime $\Leftrightarrow A/I$ is a domain (no zero divisors)

(b) $I \subset A$ is maximal $\Leftrightarrow A/I$ is a field

(c) Maximal $\Rightarrow$ prime

(d) There are prime ideals which are not maximal (later today)

Ex ① $A = \mathbb{K}[x]$ $\mathbb{K}$ = any field

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Fact: • A is a principal ideal domain

(that is, any ideal in $A = (f)$)

• Any polynomial is a product of irreducible factors in a unique way (unique factorization)

1a $I = (f)$ is prime $\Leftrightarrow$ f irreducible or 0

Pf: $\Rightarrow$ Assume $f = ab$, then $ab \in (f)$ but $a \notin (f)$, $b \notin (f)$ not prime

$f \neq 0$

$\Leftarrow$ Assume f irreducible, $ab \in (f) \Rightarrow$ ab divisible by $f \Rightarrow$ (by unique factorization) either a or b divisible by $f \Rightarrow a \in (f)$ or $b \in (f)$.

1b $I = (f)$ is maximal $\Leftrightarrow$ f irreducible

$\Rightarrow$ if $f = ab$ $(f) \subset (a) \Rightarrow$ not maximal

$\Leftarrow$ if f irreducible, $(f) \subset J = (a)$

J must be principal ideal

$\Rightarrow f$ divisible by a

[illegible]

$\Rightarrow f$ divisible by a ideal

$\Rightarrow a = f$ or $a = 1$ (up to a unit in $\mathbb{K}$)

$\Rightarrow (a) = (f)$ or $(a) = A$

$\Rightarrow (f)$ is maximal.

Note: 0 is prime but <u>not</u> maximal

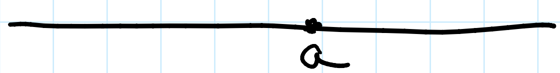
$$\operatorname{Spec} \mathbb{K}[x] = \left\{ \begin{matrix} \text{all irreducible} \\ \text{polynomials}/\mathbb{K} \end{matrix} \right\} \cup \{0\}$$

closed points — "generic point"

<u>Ex</u> $\mathbb{K} = \overline{\mathbb{K}}$ any irred. poly $= x - a$

$$\operatorname{Spec} \mathbb{K}[x] = \{(x-a)\}_{a \in \mathbb{K}} \cup \{0\}$$

{ line over $\mathbb{K}$ — $\{0\}$ not associated to any point "spread out" on the line.



<u>Ex</u>: $\mathbb{K} = \mathbb{R}$ irred. poly $=$

$$= \{ x - a, \; x^2 + ax + b \}$$

no real roots.

$$\operatorname{Spec} \mathbb{R}[x] = \underline{\mathbb{R}} \cup \{0\} \cup \{x^2 + ax + b\}$$

pairs of complex conjugate roots.

"fold $\mathbb{C}$ along the real line"

Rmk: I = maximal ideal

real line

Rmk I = maximal ideal

A/I = field = "residue field at I"

Closed point of Spec A $\rightsquigarrow$ residue field

$$\mathbb{R}[x]/(x-a) = \mathbb{R} \qquad \mathbb{R}[x]/\underset{\text{irred}}{(x^2+ax+b)} \overset{\text{exercise}}{=} \mathbb{C} \quad \text{for all } a,b.$$

functions $p(x)$ such that $p(a)=0$

$$\begin{array}{ccc} \mathbb{R}[x] & \longrightarrow & \mathbb{R} \\ p(x) & \longmapsto & p(a) = \text{value at } a \end{array}$$

Ker $= \underline{(x-a)}$ surj

$$\frac{(\text{functions on } \mathbb{R})}{(\text{functions vanishing at } a)} = \text{functions on } \{a\}$$

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② $A = \mathbb{K}[x_1 \dots x_n]$ $\mathbb{K} = \overline{\mathbb{K}}$ alg. closed –

Thm Maximal ideals in A $= \{ (x_1-a_1, x_2-a_2, \dots x_n-a_n) \}$ $a_1, \dots a_n \in \mathbb{K}$ ← n generators.

1) $(x_1-a_1, \dots x_n-a_n)$ is maximal (works for any $\mathbb{K}$)

$\mathbb{K}[x_1 \dots x_n] \longrightarrow \mathbb{K}$

$$\mathbb{K}[x_1, \dots x_n] \longrightarrow \mathbb{K}$$ (works for any $\mathbb{K}$)

$$p(x_1, \dots x_n) \longrightarrow p(a_1, \dots a_n)$$

surjective

<u>Claim</u>: $\ker = (x_1 - a_1, \dots x_n - a_n)$

Any polynomial can be written as

$$p(x_1, \dots x_n) = c_0 + \underbrace{\sum_{\substack{\text{at least} \\ \text{one } k_i > 0}} c_{k_1 \dots k_n} (x_1 - a_1)^{k_1} \dots (x_n - a_n)^{k_n}}$$

in $(x_1 - a_1, \dots x_n - a_n)$

$$p(a_1 \dots a_n) = c_0$$

$n=2$ $\quad c_0 + \dots (x_1 - a_1) + \dots (x_2 - a_2) + \dots (x_1 - a_1)^2 + \dots (x_2 - a_2)^2 + \dots (x_1 - a_1)(x_2 - a_2)$

$$\frac{\mathbb{K}[x_1 \dots x_n]}{(x_1 - a_1, \dots x_n - a_n)} \simeq \mathbb{K}$$ by Isomorphism Theorem.

$\Rightarrow$ maximal ideal.

(b) Assume that I is maximal

$V(I)$ = zero set of I in $\mathbb{K}^n$

= set of points where <u>all</u> polynomials in I vanish

<u>Nullstellensatz</u> (Hilbert) If $\mathbb{K} = \overline{\mathbb{K}}$ and $I \neq A$

then $V(I)$ is non empty, it has a point

$(a_1, \dots a_n)$

Then $V(I)$ is non empty, it has a point $(a_1 \ldots a_n)$

$\Rightarrow$ all polynomials in I vanish at $(a_1 \ldots a_n)$

$\Rightarrow I \subset (x_1 - a_1, \ldots x_n - a_n)$

Since I is maximal, $I = (x_1 - a_1, \ldots x_n - a_n)$. ∎

$\left\{ \begin{array}{l} \text{maximal ideals} \\ \text{in } \mathbb{K}[x_1 \ldots x_n] \end{array} \right\} = \{(x_1 - a_1, \ldots x_n - a_n)\}$

closed points of Spec A $\quad\quad\quad\quad \cong \mathbb{K}^n$

<u>Note</u> (f) is prime if f irreducible (still have unique factorization in $\mathbb{K}[x_1 \ldots x_n]$) but not maximal $\Rightarrow$ tons of prime ideal

<u>Ex</u> $\mathbb{C}[x, y]$ maximal ideals $\Leftrightarrow (x-a, y-b)$

$\Leftrightarrow$ point (a, b) in $\mathbb{C}^2$

$(x^2 - y^3)$ is prime since $x^2 - y^3$ irred but not maximal.

③ $\operatorname{Spec} \mathbb{Z} = \{(0), (\underset{\text{prime}}{p})\}$

same logic as $\mathbb{K}[x]$.

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same logic as $\mathbb{R}[x]$.

Remarks on Nullstellensatz

$\{x=0,\ x=1\}$ no solution!

$x-1=0$

$1 = x-(x-1) \in (x, x-1)$

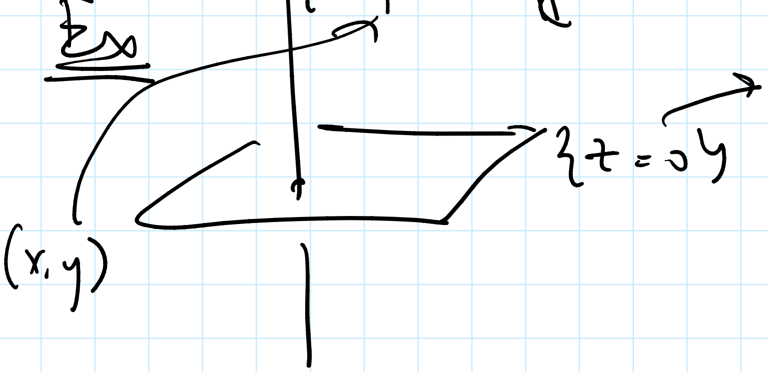
Thm $\{f_1 = f_2 \dots = f_k = 0\}$

always has a solution unless

$(f_1 \dots f_k) = \mathbb{K}[x_1 \dots x_n]$

equivalently, $1 = g_1 \cdot f_1 + \dots + g_k \cdot f_k$.

Ex $\{x=y=0\}$ $\mathbb{C}^3$



Exercise: (z) and (x,y) and
ideal of intersection $=(x,y,z)$ are all prime
max
but $(z) \cap (x,y)$ = ideal for the union
not prime.

Please do!

Exercise (3.2.J) A = ring, J = ideal

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$\phi : A \to A/J$

(a) $I \subset A/J \Longleftrightarrow \phi^{-1}(I)$ ideal in A containing J

(b) $I \subset A/J$ prime $\Longleftrightarrow \phi^{-1}(I)$ prime

(c) $I \subset A/J$ maximal $\Longleftrightarrow \phi^{-1}(I)$ maximal

Cor $A = \dfrac{\mathbb{K}[x_1, \dots, x_n]}{J}$ J = some ideal

$\operatorname{Spec} A = \left\{ \begin{matrix} \text{prime ideals} \\ \text{in } \mathbb{K}[x_1 \dots x_n] \\ \text{containing } J \end{matrix} \right\} \supset \left\{ \begin{matrix} \text{maximal ideals} \\ \text{in } \mathbb{K}[x_1 \dots x_n] \\ \text{containing } J \end{matrix} \right\}$

Ex $I = (5, p(x)) \subset \mathbb{Z}[x]$

is it prime?

$J = (5) \subset I$

3.2.5

I is prime $\Longleftrightarrow (p(x))$ is prime in $\dfrac{\mathbb{Z}[x]}{(5)} = \mathbb{Z}_5[x]$

$p(x)$ is irreducible mod 5

field