Meeting 1/21

Wednesday, January 20, 2021 10:36 PM

$\mathbb{C}[x,y,z]$

$\cup$

max

Exercise: (z) and (x,y) and ideal of intersection $=(x,y,z)$ are all prime

but $(z)\cap(x,y)$ = ideal for the union not prime.

Please do!

Exercise (3.2.J) A = ring, J = ideal

$\phi: A \to A/J$

(a) $I \subset A/J \Longleftrightarrow \phi^{-1}(I)$ ideal in A containing J

(b) $I \subset A/J$ prime $\Leftrightarrow \phi^{-1}(I)$ prime

(c) $I \subset A/J$ maximal $\Leftrightarrow \phi^{-1}(I)$ maximal

$$\begin{array}{ccc} \phi: A & \to & A/J \\ \cup & & \cup \\ \phi^{-1}(I) & & I \end{array}$$

Suppose $\phi^{-1}(I)$ is prime

$ab \in I$

Choose $\bar{a}, \bar{b} \in A$

Such that $\varphi(\bar{a}) = a$

such that $\varphi(\bar{a}) = a$

$\varphi(\bar{b}) = b$

$\varphi(\bar{a}\bar{b}) = ab \in I$

$\bar{a}\bar{b} \in \varphi^{-1}(I) \Rightarrow$ either $\bar{a} \in \varphi^{-1}(I)$

or $\bar{b} \in \varphi^{-1}(I)$

$\Rightarrow$ either $\varphi(\bar{a}) = a \in I$ or

$\varphi(\bar{b}) = b \in I$.

$$\boxed{A/\phi^{-1}(I) \simeq (A/J)/I}$$

3rd isom. theor.

Note:

$I = \phi^{-1}(I)/J$

$I \subset A/J$ max if for all I'

$\underbrace{A/J \supset I' \supset I}$ either $I' = A/J$

or $I' = I$

$\|$

$A \supset \phi^{-1}(I') \supset \underline{\phi^{-1}(I)}$

If $\phi^{-1}(I)$ is maximal then I max

$\phi^{-1}(I') = A$ or $\phi^{-1}(I') = \phi^{-1}(I)$

$\Rightarrow I' = I$.

$A \supset I'' \supset \phi^{-1}(I) \supset J$

ideal containing J

$\Longleftrightarrow$ ideal I''/J = ideal in A/J.

$I'' = \varphi^{-1}(I''/J)$.

$$I'' = \varphi^{-1}(I''/J).$$

$(x,y) \subset \mathbb{C}[x,y,z]$ prime

$\Updownarrow$ by Exercise

$(y) \subset \dfrac{\mathbb{C}[x,y,z]}{(x)}$ prime

$\dfrac{\mathbb{C}[x,y,z]}{(x)} = \mathbb{C}[y,z]$

irreducible in $\mathbb{C}[y,z]$.

Easier: Check $\dfrac{\mathbb{C}[x,y,z]}{(x,y)} \cong \mathbb{C}[z]$ is a domain

Directly: $f(x,y,z) \cdot g(x,y,z) \in (x,y)$

Any function can be written as $h = x \cdot a + y \cdot b + c(z)$

$h \in (x,y)$ iff $c(z) = 0$

$f = x \cdot a + y \cdot b + c(z)$

$g = x \cdot a' + y \cdot b' + c'(z)$

$fg = \underbrace{\quad\quad}_{\in} + cc'$

$\in (x,y)$

(x,y) iff $cc' = 0$

$\Rightarrow$ either $c = 0$

or $c' = 0$.

A = ring $\quad \operatorname{Spec} A = \{\text{prime ideals in } A\} \supset \{\text{maximal ideals}\}$

$\quad$ = points $\quad\quad$ = closed .

points   closed points.

Main example: $A = \dfrac{\mathbb{K}[x_1 \dots x_n]}{J}$, $\mathbb{K} = \overline{\mathbb{K}}$

Note: Any finitely generated algebra over $\mathbb{K}$ can be written in this form.

Proof A generated by $a_1 \dots a_n$

$$\mathbb{K}[x_1 \dots x_n] \longrightarrow A$$
$$x_i \longrightarrow a_i \quad \text{homomorphism}$$

Surjective, since a_i generate A

J = Kernel of this homomorphism

$$\Rightarrow A \simeq \frac{\mathbb{K}[x_1 \dots x_n]}{J}.$$

Note This depends on the choice of generators in A, for example, $\mathbb{K}[z] \cong \dfrac{\mathbb{K}[x, y, z]}{(x, y)}$.

By exercise 3.2.J above:

$$\left\{ \begin{array}{c}\text{prime ideals} \\ \text{in } \dfrac{\mathbb{K}[x_1 \dots x_n]}{J}\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c}\text{prime ideals} \\ \text{in } \mathbb{K}[x_1 \dots x_n] \\ \text{containing } J\end{array} \right\}$$

$$\left\{ \begin{array}{c}\text{maximal ideals} \\ \text{in } \mathbb{K}[x_1 \dots x_n]\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c}\text{maximal ideals} \\ \text{in } \mathbb{K}[x_1 \dots x_n]\end{array} \right\}$$

$$\left\{\begin{matrix}\text{maximal ideals}\\ \text{in } \mathbb{K}[x_1,\dots x_n]/J\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals}\\ \text{in } \mathbb{K}[x_1,\dots x_n]\\ \text{containing } J\end{matrix}\right\}$$

$\Vert$ assuming $\mathbb{K}=\overline{\mathbb{K}}$

$$\left\{\begin{matrix}\text{maximal ideals}\\ (x_1-a_1,\dots x_n-a_n)\supset J\end{matrix}\right\}$$

$\Vert$

"classical" $\left\{\begin{matrix}\text{points } (a_1,\dots a_n)\in\mathbb{K}^n\\ \text{where } \underline{\text{all}} \text{ functions from } J\\ \text{vanish}\end{matrix}\right\}$.

Conclusion: We can describe the vanishing locus for the ideal J intrinsically just in terms of the algebra $\mathbb{K}[x_1,\dots x_n]/J$.

Ex $\mathbb{K}[z] = \dfrac{\mathbb{K}[x,y,z]}{(x,y)}$ $\mathbb{K}=\overline{\mathbb{K}}$

$$\left\{\begin{matrix}\text{maximal}\\ \text{ideals in } \mathbb{K}[z]\end{matrix}\right\} \longleftrightarrow \{(z-a),\ a\in\mathbb{K}\} = \mathbb{K} \text{ line}$$

$\updownarrow$

$$\left\{\begin{matrix}\text{maximal ideals}\\ \text{in } \mathbb{K}[x,y,z]/(x,y)\end{matrix}\right\}$$

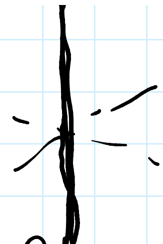
$\updownarrow$

$$\left\{\begin{matrix}\text{maximal ideals}\\ \text{in } \mathbb{K}[x,y,z] \text{ contains } (x,y)\end{matrix}\right\}$$

$\updownarrow$

$\{(x,y,z-a)\}$ = points in the line $\{x=y=0\}\subset\mathbb{K}^3$

$\mathbb{K}^3$



$\mathbb{k}^3$ | line $\{x=y=0\} \subset \mathbb{k}^3$

This is a special case of important general fact:

Thm $\varphi : A \to B$ ring homomorphism

$\varphi^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

Note: goes in the opposite way

$I \subset B$ prime ideal $\longrightarrow \varphi^{-1}(I)$ prime ideal in A

If φ is an isomorphism, φ^* is a bijection between $\operatorname{Spec} B$ and $\operatorname{Spec} A$.

Proof: $ab \in \varphi^{-1}(I) \Rightarrow \varphi(a)\varphi(b) = \varphi(ab) \in I$

Since I is prime, either $\varphi(a)$ or $\varphi(b) \in I$

$\Rightarrow$ either a or b is in I. ∎.

Ex: $\varphi : \underbrace{\mathbb{k}[x_1 \ldots x_n]}_{A} \longrightarrow \underbrace{\mathbb{k}[y_1 \ldots y_m]}_{B}$

ring homomorphism

determined by n polynomials

$\varphi_1 = \varphi(x_1), \varphi_2 = \varphi(x_2) \ldots \varphi_n = \varphi(x_n) \in \mathbb{k}[y_1 \ldots y_m]$

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{k}[y_1 \ldots y_m] & \longrightarrow & \operatorname{Spec} \mathbb{k}[x_1 \ldots x_n] \\ \| & & \| \\ \mathbb{k}^m & & \mathbb{k}^n \end{array}$$

$$(a_1 \ldots a_m) \longrightarrow (\varphi_1(a_1 \ldots a_m), \varphi_2(a_1 \ldots a_m), \ldots \varphi_n(a_1 \ldots a_m))$$

"algebraic maps from $\mathbb{k}^m$ to $\mathbb{k}^n$"

$\varphi : \mathbb{C}[x, y, z] \longrightarrow \mathbb{C}[t, u]$ homomorphism.

$$\varphi(x) = t^2 + u = \varphi_1$$

$$\varphi(y) = t + u = \varphi_2$$

$$\varphi(z) = u - t = \varphi_3$$

$$\varphi(x^a y^b z^c) = (t^2+u)^a (t+u)^b (u-t)^c$$

This should correspond to a function

$$f : \mathbb{C}^2_{t,u} \longrightarrow \mathbb{C}^3_{x,y,z}$$

$\qquad \operatorname{Spec} \mathbb{C}[t, u]$

$$\mathbb{C}^2 \ni (a, b) \longrightarrow (\varphi_1(a,b), \varphi_2(a,b), \varphi_3(a,b))$$

$$= (a^2+b, a+b, a-b).$$

maximal ideal

$$(t-a, u-b) \longleftrightarrow \varphi^{-1}(t-a, u-b)$$

$\mathbb{C}^2[t,u]$

$$= (x - (a^2+b), y - (a+b), z - (a-b)).$$

$z - (a-b)).$

$$\varphi(x - (a^2+b)) = \varphi(x) - (a^2+b)$$
$$= (t^2+u) - (a^2+b)$$

polynomial in t, u which vanishes at (a,b)

$\Rightarrow$ in $(t-a, u-b)$.

Exercise: $\operatorname{Spec} \mathbb{K}[x, x^{-1}] = ?$