Meeting 1/28

Thursday, January 28, 2021 11:01 AM

Zariski topology in $\mathbb{C}^n$

Reminder: a top. space X is a set with a collection of open & closed subsets such that:

- U is open $\longleftrightarrow$ $X \setminus U$ closed

- $\emptyset, X$ open
- Any union of open subsets is open
- Finite intersection of open subsets is open

- $\emptyset, X$ closed
- Any intersection of closed subsets is closed
- Finite union of closed subsets is closed

Def (Zariski topology) I = ideal in $\mathbb{C}[x_1 \ldots x_n]$

$V(I)$ = zero set of I (all polynomials in I vanish)

$V(I)$ closed in Zariski topology.

Thm This is a top. space

Thm This is a top. space

Proof: $\emptyset = \{1=0\} = V(\mathbb{C}[x_1 \ldots x_n])$ $\quad \mathbb{C}^n = V(0)$

- I_α = any family of ideals

$$\bigcap_\alpha V(I_\alpha) = V\Big(\underbrace{\sum_\alpha I_\alpha}_{\text{ideal}}\Big)$$

$$\sum_\alpha I_\alpha = \{ f_{\alpha_1} + \ldots + f_{\alpha_k} \}$$
$$f_{\alpha_i} \in I_{\alpha_i}$$

f_{α_i} vanishes on $V(I_{\alpha_i})$

$\Rightarrow$ vanishes on $\bigcap_\alpha V(I_\alpha)$

$\Rightarrow$ all f_{α_i} vanish on $\bigcap V(I_\alpha)$

$\Rightarrow f_{\alpha_1} + \ldots + f_{\alpha_k}$ also vanishes.

Conversely, if $p \in V(\sum_\alpha I_\alpha)$

all function from $\sum_\alpha I_\alpha$ vanish at p

$\Rightarrow$ all functions from I_α vanish at p

$\Rightarrow p \in V(I_\alpha)$ for all α

$\Rightarrow p \in V(I_\alpha)$ for all α. at p

- $V(I_1) \cup V(I_2) = V(I_1 I_2)$ Exercise.

Very weird topology:

Ex $\mathbb{C}$ all ideals $= (f)$

$$V(f) = \begin{cases} \mathbb{C}, & f = 0 \\ \emptyset, & f = \text{const} \\ n \text{ points}, & \deg f = n \end{cases}$$

$\{f = 0\}$

All closed subsets = either $\mathbb{C}$ or finite subset.

Any finite subset is closed (find a polynomial with given roots).

Open = complements to finite subsets.

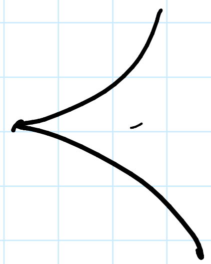
($\Leftrightarrow$ empty or dense in $\mathbb{C}$ in "usual" topology).

$\{f \neq 0\}$

Rmk: Not Hausdorff: no disjoint

RMK: not Hausdorff; no disjoint open subsets! Any two nonempty open subsets intersect.

Ex: In $\mathbb{C}^2$, $\{x^2=y^3\}$ closed



$\{x^2 \neq y^3\}$ open

In $\mathbb{C}^2$, complement to $\{f=0\}$ is open

$(0,0) = V(x,y) = \{x=0\} \cap \{y=0\}$

Complement to $(0,0) = \{x \neq 0\} \cup \{y \neq 0\}$

open cover of the complement $\mathbb{C}^2 - \{(0,0)\}$.

Fact: Zariski closed subset in $\mathbb{C}^n$ $\Longrightarrow$ closed in "usual" topology

Zariski open subset in $\mathbb{C}^n$ $\Longrightarrow$ open and dense in "usual" topology (or empty).

Caution: Zariski topology on $\mathbb{C}^2$ is NOT the product topology.

is NOT the product topology.
$\mathbb{C}\times\mathbb{C}$

<u>Thm</u>: $f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ algebraic function

$f : (a_1, \dots, a_n) \longrightarrow (\varphi_1(a_1 \dots a_n), \dots, \varphi_m(a_1 \dots a_n))$

φ_i = some polynomials in n variables.

$\Updownarrow$

$f^* : \mathbb{C}[y_1 \dots y_m] \longrightarrow \mathbb{C}[x_1 \dots x_n]$ ring homomorphism

"pullback" $\quad f^* y_i = \varphi_i(x_1 \dots x_n)$

Last time: $f^* : A \to B \rightsquigarrow f : \operatorname{Spec} B \to \operatorname{Spec} A$.

<u>Claim</u>: f is continuous in Zariski topology.

<u>Recall</u>: f is continuous $\iff$ $f^{-1}(\text{open subset})$ = open

$\iff f^{-1}(\text{closed subset})$ = closed.

$f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$

closed subset of $\mathbb{C}^m = V(I)$

I = ideal in $\mathbb{C}[y_1 \dots y_m]$

$f^{-1}(V(I)) = \{ p \in \mathbb{C}^n : f(p) \in V(I) \}$

$= \{ p :$ all polynomials $g \in I \}$

$= \{ p :$ all polynomials $g \in I$ in vanish at $f(p) \}$

$= \{ p : g(\varphi_1(p_1 \ldots p_n), \varphi_2(p_1 \ldots p_n), \ldots \varphi_m(p_1 \ldots p_n)) = 0 \}$

$f^*(g) = g(\varphi_1 \ldots \varphi_m)$

$\{ p : f^*(g)$ vanishes at $p \}$

Conclusion: $f^{-1}(V(I)) = V(f^*(I))$

$\Rightarrow f^{-1}$ (closed) is closed.

<u>Zariski topology for Spec A</u>

A = ring

Spec A = $\{$ prime ideals in $A \}$

I = ideal in A

$V(I) = \{$ prime ideals $\mathfrak{p} \supset I \}$

Zariski topology: $V(I)$ are closed

<u>Note:</u> $A = \mathbb{C}[x_1 \ldots x_n]$

$\mathfrak{p} = (x_1 - a_1, \ldots x_n - a_n) \leftarrow$ maximal ideal

$I \subset \mathfrak{p} \Leftrightarrow I \subset (x_1 - a_1, \ldots x_n - a_n)$

$I \subset \mathfrak{p} \iff I \subset (x_1 - a_1, \dots, x_n - a_n)$

$\iff$ all functions in I vanish at $(a_1, \dots, a_n)$

<u>Thm</u> (a) Spec A is a topological space with Zariski topology

(b) $f^*: A \longrightarrow B$ ring homomorphism

f: Spec $B \longrightarrow$ Spec A continuous function in Zariski topology

---

<u>Proof:</u> (a) $\emptyset = V(A)$ $\qquad$ Spec $A = V(0)$

$\mathfrak{p} \supset A \Rightarrow \mathfrak{p} = A$ but $\mathfrak{p}$ proper

Arbitrary intersections

$$\bigcap_\alpha V(I_\alpha) = \bigcap_\alpha \{\mathfrak{p} \supset I_\alpha\}$$

$$= \{\mathfrak{p} \supset \sum_\alpha I_\alpha\}$$

Finite unions $\quad V(I_1) \cup V(I_2) = V(I_1 I_2)$

$\{\mathfrak{p}_{\text{prime}} \supset I_1\}$ or $\{\mathfrak{p} \supset I_2\}$

prime

$I_1 I_2$ = ideal generated by fg $f \in I_1$, $g \in I_2$

- If $\mathfrak{p} \supset I_1$ then $f \in \mathfrak{p}$, since $\mathfrak{p}$ is an ideal, $fg \in \mathfrak{p}$ $\Rightarrow$ all such $fg \in \mathfrak{p}$
- If $fg \in \mathfrak{p}$, since $\mathfrak{p}$ is prime, either f or $g \in \mathfrak{p}$ $\Rightarrow I_1 \subset \mathfrak{p}$ or $I_2 \subset \mathfrak{p}$.

Allows to define topology on $\operatorname{Spec} \mathbb{Z}$, $\operatorname{Spec} \mathbb{k}[x]$

$\mathbb{k}$ = any field

(b) $f^*: A \longrightarrow B$

$f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

$\cup$

$V(I)$ closed

$f^{-1}(V(I)) = \{ \text{prime ideals } \mathfrak{p} \text{ in } B \mid f(\mathfrak{p}) \supset I \}$

$//$

$f^*(\mathfrak{p}) \supset f^*(I) \Longleftrightarrow f^{*-1}(\mathfrak{p}) \supset I$

Claim: $f^{-1}(V(I)) = V(f^*(I))$

$\varphi^*(V(I)) = V(\varphi^*(I))$

closed in Spec B.

Ex $A = \dfrac{\mathbb{C}[x_1 \ldots x_n]}{J}$

Spec $A \supset$ maximal ideals in A = maximal ideals in $\mathbb{C}[x_1 \ldots x_n]$ containing J

$= V(J) =$ zero set of J.

I = ideal in $\dfrac{\mathbb{C}[x_1 \ldots x_n]}{J} \longleftrightarrow \tilde{I}$ ideal in $\mathbb{C}[x_1 \ldots x_n]$ containing J

$$V(I) = V(\tilde{I}) \subset V(J)$$

Zariski closed subsets of $V(J)$

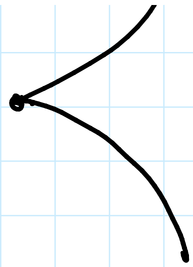
= intersections of closed subsets in $\mathbb{C}^n$ with $V(J)$

= closed subsets in $\mathbb{C}^n$ contained in $V(J)$.

Ex $J = (x^2 - y^3)$

$V(J) \subset \mathbb{C}^2$

Zariski closed subsets of $\{x^2 = y^3\}$



Zariski closed subsets of $\{x^2=y^3\}$

$\Updownarrow$

$V(I)$ for ideal I contains x^2-y^3.

$\boxed{\text{Hw: } \operatorname{Spec} \mathbb{K}[x,x^{-1}]}$