Meeting 1/7

Thursday, January 7, 2021 10:50 AM

$\mathbb{K}$ = field $\mathbb{K}^n$ = n-dim vector space

$$X = \{ f_1(x_1 \ldots x_n) = \ldots = f_k(x_1 \ldots x_n) = 0 \}$$

$f_1 \ldots f_k$ = polynomials

X = subset of $\mathbb{K}^n$ cut out by polynomial equations = affine algebraic variety

$\mathbb{K}[x_1 \ldots x_n]$ = polynomials in n vars = algebraic functions on $\mathbb{K}^n$

I_X = ideal in $\mathbb{K}[x_1 \ldots x_n]$ generated by $f_1 \ldots f_k$

$$= \{ g_1 f_1 + g_2 f_2 + \ldots + g_k f_k \}$$

any function in I_X vanishes on X

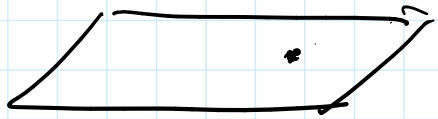
$X = V(I_X)$ = zero set of the ideal

$$\mathcal{O}_X = \frac{\mathbb{K}[x_1 \ldots x_n]}{I_X} = \frac{\mathbb{K}[x_1 \ldots x_n]}{(f_1 \ldots f_k)}$$ ring of algebraic functions on X

Can define the value of any function in $\mathcal{O}_X$

Main question: how the geometric properties of X are related to algebraic properties of X are related to algebraic properties of I_X or $\mathcal{O}_X$?

<u>Ex</u> $\mathbb{K}^3$

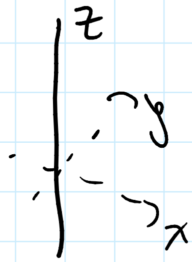
X  $\{z=0\}$

$I_X = (z)$

$$\mathcal{O}_X = \frac{\mathbb{K}[x,y,z]}{(z)} = \mathbb{K}[x,y] = \text{functions on } X$$

$p(x,y,z) + zh$ eval at $(x,y,0)$

value does not depend on h

$Y =$  $\{x=0, y=0\}$

$I_Y = (x,y) = \{ax + by\}$

$$\mathcal{O}_Y = \frac{\mathbb{K}[x,y,z]}{(x,y)} = \mathbb{K}[z].$$

<u>Easy question</u> If X, Y are defined by some equations/ideals, how to think of $X \cap Y$, $X \cup Y$, $X \times Y$?

(a) $X = \{f_1 = \dots = f_k = 0\}$ $\quad X \cap Y = \{f_1 = \dots = f_k = \dots = g_\ell = 0\}$

$Y = \{g_1 = \dots = g_\ell = 0\}$ $\quad \boxed{I_{X\cap Y} = I_X + I_Y}$

In our example: $X \cap Y = \{x = y = z = 0\}$

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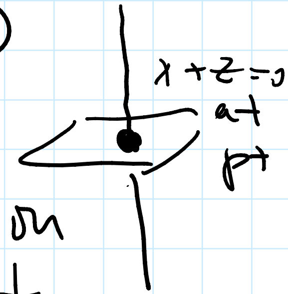
$= \text{point } (0,0,0)$

$I_{X \cap Y} = (x,y) + (z) = (x,y,z)$



$\mathcal{O}_{X \cap Y} = \dfrac{\mathbb{K}[x,y,z]}{(x,y,z)} = \mathbb{K}$ = functions on the point

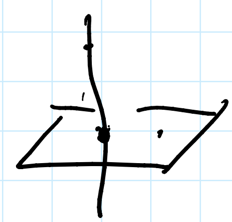
(b) $X \cup Y \rightsquigarrow \boxed{I_{X \cup Y} = I_X \cap I_Y}$

A function vanishes on $X \cup Y$ iff it vanishes on X AND on Y

<u>Exercise</u>: $I_{X \cup Y} = (x,y) \cap (z) = \underline{(xz, yz)}$

xz vanishes on both the line and the plane!

$\{xz = yz = 0\}$ defines the union of the line & plane.



(c) $X \times Y$ In the example:

$$X \times Y \subset \underset{x_1,y_1,z_1}{\mathbb{K}^3} \times \underset{x_2,y_2,z_2}{\mathbb{K}^3} = \mathbb{K}^6$$

Equs $z_1 = 0$, $x_2 = y_2 = 0$

X in the first copy of $\mathbb{K}^3$ $\qquad$ Y in the second copy of $\mathbb{K}^3$

$\mathcal{O} = \mathbb{K}[x_1, y_1, z_1, x_2, y_2, z_2]$

$$\mathcal{O}_{X\times Y} = \frac{\mathbb{k}[x_1,y_1,z_1,x_2,y_2,z_2]}{(z_1,x_2,y_2)} = \mathcal{O}_X \otimes_{\mathbb{k}} \mathcal{O}_Y$$

In general, $\boxed{\mathcal{O}_{X\times Y} = \mathcal{O}_X \otimes_{\mathbb{k}} \mathcal{O}_Y}$ exercise

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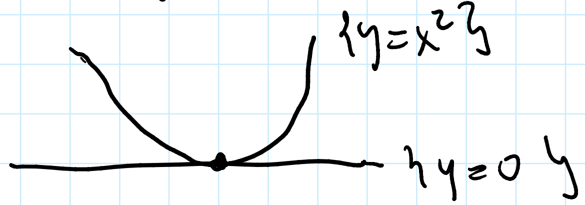
Harder questions: • $\dim X$? is X smooth? $(\mathbb{k}=\mathbb{R},\mathbb{C})$

- What does it mean to be smooth over $\mathbb{k}$?
- How to understand geometry/topology of X from $\mathcal{O}_X$
- For finite $\mathbb{k}$, how many points are there in X? $\mathbb{k}=\mathbb{Z}_p$ (count solutions $f_1=\dots=f_k=0 \mod p$)
- Maps $X \longrightarrow Y$
- When X and Y are "the same" (isomorphic)?

Issues: (1) X could be empty!

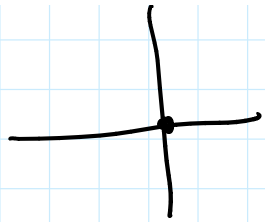
$$\{x^2+y^2+1=0\} \subset \mathbb{R}^2$$

$$\{x^p+y^p=z^p\} \subset \mathbb{Q}^3 \leftarrow \text{Fermat's theorem}$$

(2) 

$$\mathcal{O}_{\{y=0\}\cap\{y=x^2\}} = \frac{\mathbb{k}[x,y]}{(y-x^2,y)} = \frac{\mathbb{k}[x,y]}{(y,x^2)} = \frac{\mathbb{k}[x]}{(x^2)}$$

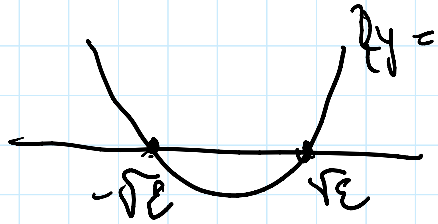
$$\frac{}{(y, x^2)} = \frac{}{(x^2)}$$
dim = 2



$$\frac{\mathbb{k}[x,y]}{(x,y)} = \frac{\mathbb{k}[x]}{(x)} = \mathbb{k}$$
dim = 1

Same point, different algebras!

<u>Note:</u> dim = 2 "captures tangency"



$$\mathcal{O} = \frac{\mathbb{k}[x]}{(x^2-\varepsilon)} \simeq \frac{\mathbb{k}[x]}{(x-\sqrt{\varepsilon})} \oplus \frac{\mathbb{k}[x]}{(x+\sqrt{\varepsilon})}$$

<u>Exercise</u>

$$\dim \frac{\mathbb{k}[x]}{(f)} = n \qquad f = \text{polynomial of degree } n$$

- Ex: functions on n distinct roots
- Also: roots with multiplicity

(3) Can consider more general "fat points"

$\{x^2 = y^2 = xy = 0\}$ defines $X = (0,0)$

$$\dim = \frac{\mathbb{k}[x,y]}{(x^2, y^2, xy)} \overset{\text{exercise}}{=} 3$$

<u>Solution to issues (2), (3):</u> "scheme structure"

affine scheme $\approx$ pair (X = alg. variety, $\mathcal{O}_X$ = alg. of functions) affine scheme $\approx$ pair (X = alg. variety, $\mathcal{O}_X$ = alg. of functions)

Could have the same set X with different "scheme structure" = different $\mathcal{O}_X$

<u>Solution to (1)</u>: change the field from $\mathbb{K}$ to $\overline{\mathbb{K}}$ = algebraic closure.

<u>Hilbert's Nullstellensatz</u>: If $\mathbb{K} = \overline{\mathbb{K}}$ alg. closed, and $I \subsetneq$ = proper ideal in $\mathbb{K}[x_1, \dots, x_n]$

Then the zero set of $I = V(I) \neq \emptyset$

X has a point!

<u>Ex</u> $\{x^2 + y^2 + 1 = 0\} \subset \mathbb{C}^2$

is interesting, non-empty

Galois group of $\mathbb{C}/\mathbb{R}$ $= \mathbb{Z}_2$ acts by complex conjugation

$(x, y) \longrightarrow (\bar{x}, \bar{y})$

fixed points of $\mathbb{Z}_2 = \{ \cdots \} \subset \mathbb{R}^2$