Meeting 3/11

Wednesday, March 10, 2021 6:09 PM

Def A ringed space is a top. space X with a sheaf of rings $\mathcal{O}_X$

ring of functions on U

Unpack: • Open $U \longrightarrow \mathcal{O}_X(U)$ ring

• $V \subset U \longrightarrow$ restriction map

$$\text{res}: \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$$

ring homomorphism.

Ex $\mathcal{O}_X(U)$ = ring of functions on U

$\text{res}(f)$ = restriction of f (defined on U) to V

$$\text{res}(f \cdot g) = \text{res}(f) \cdot \text{res}(g)$$

for all x in V, $f(x) \cdot g(x) = \text{res}(f \cdot g)(x)$

$\mathcal{O}_X$ satisfies axioms of a sheaf:

$\text{res}_{U,U} = \text{id}$ •
$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r] \arrow[rr, bend right] & \mathcal{O}_X(V) \arrow[r] & \mathcal{O}_X(W) \end{tikzcd}$$

Gluing: $V = \bigcup_i V_i$ $f_i \in \mathcal{O}_X(V_i)$ such that

$$f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j} \text{ for all } i, j$$

then $\exists f \in \mathcal{O}_X(V)$ such that $f|_{V_i} = f_i$

$\exists\, f \in \mathcal{O}_X(U)$ such that $f|_{V_i} = f_i$

Identity: such f is unique.

Def A scheme is a ringed space $(X, \mathcal{O}_X)$ such that $X = \bigcup_i V_i$ where V_i open and $(V_i, \mathcal{O}_X(V_i))$ is an affine scheme w. Zariski topology. That is

$$\boxed{V_i = \operatorname{Spec}(A_i) \qquad A_i = \mathcal{O}_X(V_i) = \text{ring}.}$$

V_i = affine open cover of X

X "locally" looks like $\operatorname{Spec} A$

Each point of X has a neighborhood (one of V_i) which looks like $\operatorname{Spec} A_i$.

Sheaf machinery is necessary to glue these local pieces together.

Ex $X = \mathbb{P}^n$, V_i = charts $\{x_i \neq 0\} \simeq \operatorname{Spec} \mathbb{C}[x_1 \ldots x_n]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbb{K}^n$

Q: Given some open subset $U \subset X$ how to define $\mathcal{O}_X(U)$?

- U is covered by $U \cap V_i$ ← open

- U is covered by $U \cap V_i$ + open
it is sufficient to understand
$\mathcal{O}_X(U \cap V_i)$ + gluing
- $U \cap V_i$ = Zariski open in $V_i = \operatorname{Spec} A_i$
covered by distinguished open subsets

$$\underset{\operatorname{Spec} A_i}{D(f) \subset} = \{f \neq 0\} = \{\text{prime ideals not containing } f\}$$
$f \in A_i$

Sufficient to understand these (+ gluing)

- $$\mathcal{O}_X(D(f)) = A_i[f^{-1}] = \left\{ \frac{a}{f^k} : \begin{array}{l} a \in A_i \\ k \geq 0 \end{array} \right\}$$

localization

To summarize, we cover U by distinguished open subsets which are contained in one of the charts V_i and have the form $D(f)$.

Very good exercise: this is consistent and defines a sheaf on X (see Theorem 4.1.2 in Vakil).

Example $\mathbb{C}^2 \setminus \{(0,0)\} = X$

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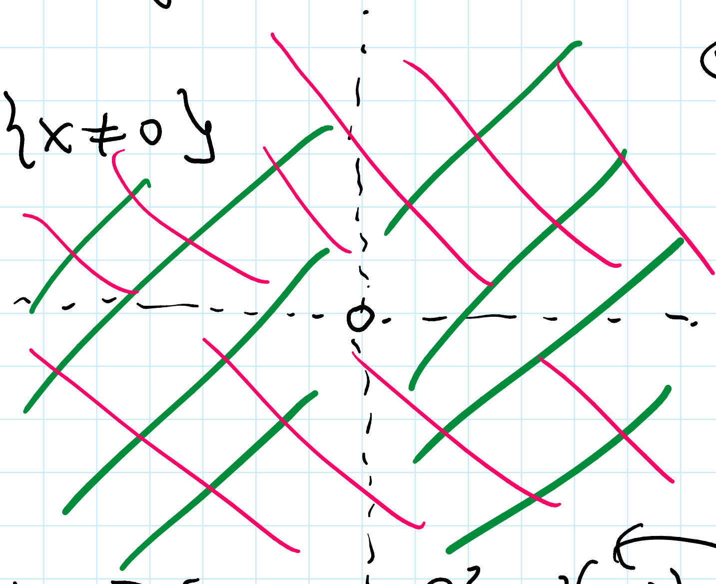
We want to understand the sheaf of functions on X.

$D(x) = \{x \neq 0\}$

$\mathbb{C}^2$



$D(y) = \{y \neq 0\}$

$D(x) \cup D(y) = \mathbb{C}^2 \setminus \{(0,0)\}$ either $x \neq 0$ or $y \neq 0$

$D(x) \cap D(y) = \{x \neq 0 \text{ AND } y \neq 0\}$

$= \{xy \neq 0\} = D(xy)$

$\mathcal{O}(D(x)) = $ functions on $D(x)$

$= \left\{ \dfrac{a(x,y)}{x^k} \right\}$ polynomial in x, y. can divide by x where $x \neq 0$

$\mathcal{O}(D(y)) = $ functions on $D(y)$

$= \left\{ \dfrac{b(x,y)}{y^k} \right\}$ can divide by y where $y \neq 0$

Q: How to describe functions on $\mathbb{C}^2 \setminus \{(0,0)\}$

A: These are the same as pairs (f_1, f_2)

A. These are the same as pairs (f_1, f_2)

f_1 = function on $D(x)$ f_2 = function on $D(y)$

f_1 and f_2 agree on $D(x) \cap D(y)$.

<u>Explicitly</u>: $f_1 = \frac{a(x,y)}{x^k}$ $f_2 = \frac{b(x,y)}{y^k}$

$$\frac{a(x,y)}{x^k} = \frac{b(x,y)}{y^k}$$ whenever $x \neq 0$ and $y \neq 0$

$$y^k a(x,y) = x^k b(x,y)$$

<u>Unique factorization</u>, x, y do not have common factors $\Rightarrow a(x,y)$ divisible by x^k

$b(x,y)$ divisible by y^k.

$\Rightarrow$ function $\frac{a(x,y)}{x^k} = \frac{b(x,y)}{y^k} = c(x,y)$ polynomial in $\mathbb{C}[x,y]$

$$\boxed{\text{Conclusion: Algebraic functions on } \mathbb{C}^2 \setminus \{(0,0)\} = \mathbb{C}[x,y]}.$$

over alg. closed field

<u>Cor.</u>: Any function defined outside of $(0,0)$ can be extended to $(0,0)$.

extension to codim 2

≈ Hartogs thm in complex analysis

$\approx$ Hartogs thm in complex analysis. to codim 2

Cor $\mathbb{C}^2 \setminus \{(0,0)\}$ is not an affine scheme.

Proof Assume that $\mathbb{C}^2 \setminus \{(0,0)\} = \operatorname{Spec} A$ for some A, then $A =$ ring of functions on $\mathbb{C}^2 \setminus \{0,0\}$ $\Rightarrow A = \mathbb{C}[x,y]$, but $\operatorname{Spec} \mathbb{C}[x,y] = \mathbb{C}^2$. Contradiction. ∎

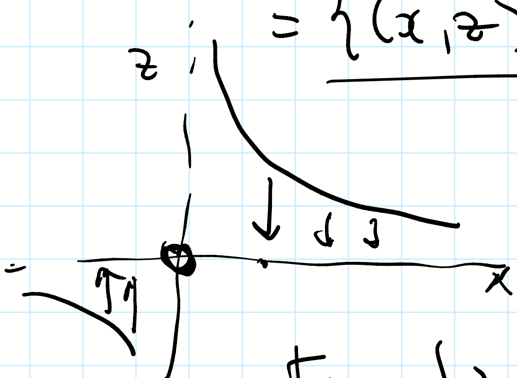
Reminder: $X = \operatorname{Spec} A$ (ex. $\mathbb{C}^2$ (Zariski))

$D(f) = \{f \neq 0\} =$ open subset in X

$= \{$prime ideals not containing $f\}$

$D(f) =$ closed subset in $\underbrace{X \times \mathbb{C}_z}$

$= \{(x,z) : f(x) \cdot z = 1\}$.



$z = \frac{1}{f(x)}$

$D(x) = \{x \neq 0\} = \{xz = 1\}$

Functions on $X \times \mathbb{C}_z = A[z] =$ polynomials in z w. coefs in A

function on X

$= A \otimes_{\mathbb{C}} \mathbb{C}[z]$

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Functions on $\{f(x)\cdot z = 1\} \subset X \times \mathbb{C}_z$

$$= \frac{A[z]}{(f(x)\cdot z - 1)} = A\left[\frac{1}{f(x)}\right] = A[f^{-1}] = \underline{\underline{\mathcal{O}(D(f))}}.$$

ideal defining $\{f(x)\cdot z - 1 = 0\}$

Functions on $\{xz = 1\}$ $= \frac{\mathbb{C}[x,z]}{(xz-1)} = \mathbb{C}[x, x^{-1}] = \mathcal{O}(D(x))$

$$\frac{a}{f^k} \longleftrightarrow \underbrace{a\cdot z^k}$$

<u>Exercise/Fact</u> Prime ideals in $A[f^{-1}]$ = prime ideals in A not containing f.

X = any scheme
always have a map.

$X \longrightarrow \operatorname{Spec} \mathcal{O}_X$

$\mathcal{O}_X \longleftarrow$ functions on ~~Spec~~ $\mathcal{O}_X$ = global functions on X

$V_i = \operatorname{Spec} A_i$ global function $\underline{\operatorname{Spec}}$ $\{$global

$V_i = \operatorname{Spec} A_i$ global function $\operatorname{Spec}\{\text{global functions}\}$

for all i $\quad \downarrow \quad A_i \qquad \operatorname{Spec} A_i \to \operatorname{Spec}\{\text{global functions}\}$

<u>Ex</u> $X = \mathbb{P}^n$ Global fns $= \mathbb{C}$

$$\mathbb{P}^n \longrightarrow \operatorname{Spec} \mathbb{C} = \{pt\}.$$