Meeting 3/4

Thursday, March 4, 2021 10:28 AM

Sheaves

X = top. space

Def A sheaf $\mathcal{F}$ on X is the following data

- For each open subset $U \subset X$, an object $\mathcal{F}(U)$

v. space
ring
module
group

- For each inclusion $V \hookrightarrow U$ a restriction map

ring homomorphism
module

$$\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

Satisfying the following conditions:

presheaf

(1) $\mathrm{res}_{U,U} = \mathrm{id} \quad \mathcal{F}(U) \to \mathcal{F}(U)$

(2) $U \hookrightarrow V \hookrightarrow W$

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\mathrm{res}_{W,V}"] \arrow[rr, bend right, "\mathrm{res}_{W,U}"'] & \mathcal{F}(V) \arrow[r, "\mathrm{res}_{V,U}"] & \mathcal{F}(U) \end{tikzcd}$$

$$\mathrm{res}_{W,U} = \mathrm{res}_{V,U}\, \mathrm{res}_{W,V}$$

union

(3) (Identity) Suppose $V = \bigcup_i V_i$ open

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$f, g \in \mathcal{F}(V)$ If we know that

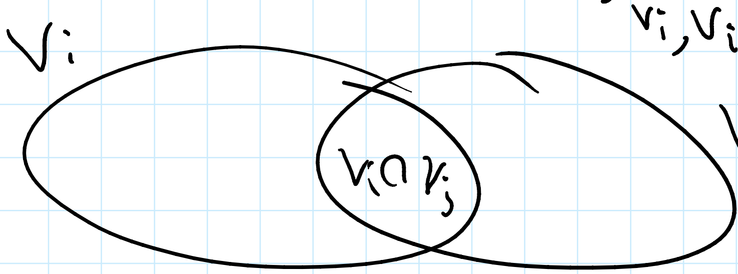
$res_{V,V_i}(f) = res_{V,V_i}(g)$

Then $f = g$

(4) (Gluing) Suppose $V = \bigcup_i V_i$

We are given a collection $f_i \in \mathcal{F}(V_i)$

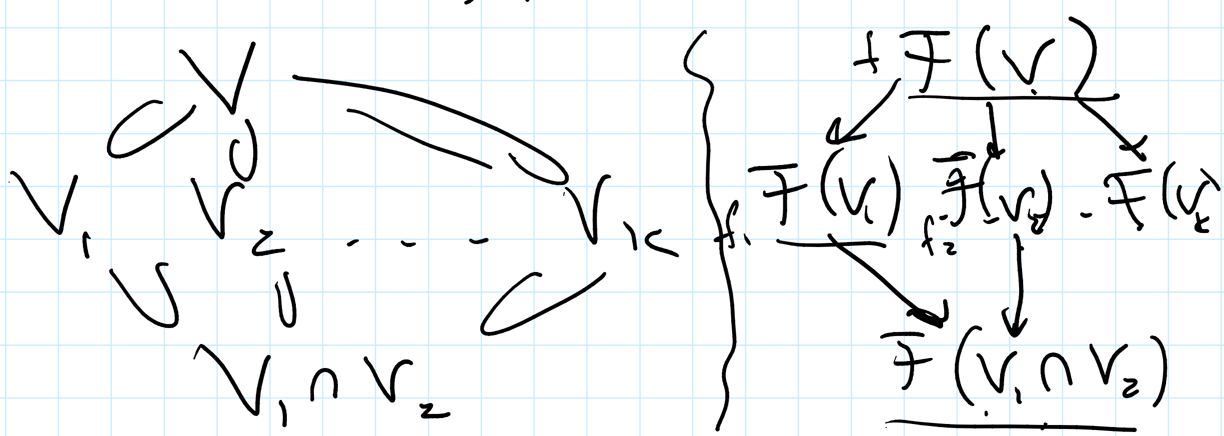
such that $res_{V_i, V_i \cap V_j} f_i = res_{V_j, V_i \cap V_j} f_j$



for each i, j

__Then__ there exists $f \in \mathcal{F}(V)$

such that $res_{V,V_i} f = f_i$.



By (2), the condition that

$res_{V_i, V_i \cap V_j} f_i = res_{V_j, V_i \cap V_j} f_j$

$$\mathrm{res}_{V_i, V_i \cap V_j} f_i = \mathrm{res}_{V_j, V_i \cap V_j} f_j$$

is <u>necessary</u> for (4) since

$$f_i = \mathrm{res}_{V, V_i} f \implies \mathrm{res}_{V_i, V_i \cap V_j} f_i = \mathrm{res}_{V, V_i \cap V_j} f = \mathrm{res}_{V_j, V_i \cap V_j} f_j$$

Gluing axiom says that it is sufficient, and by identity axiom gluing is unique.

<u>Ex</u> $\mathcal{F}(U) = \{$all continuous functions on U (in normal topology)$\}$

$V \subset U$ open $\quad f$ continuous on U

$\implies$ continuous on V

$$\mathcal{F}(U) \xrightarrow{\mathrm{res}} \mathcal{F}(V)$$

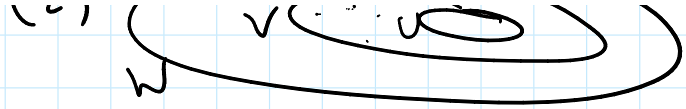
continuous function on U $\qquad$ continuous functions on V

<u>Check the axioms:</u> (1) $\mathrm{res}_{U,U} = \mathrm{id}$ clear

$V = U \quad \mathrm{res}_{U,U}(f) =$ same function

(2) 

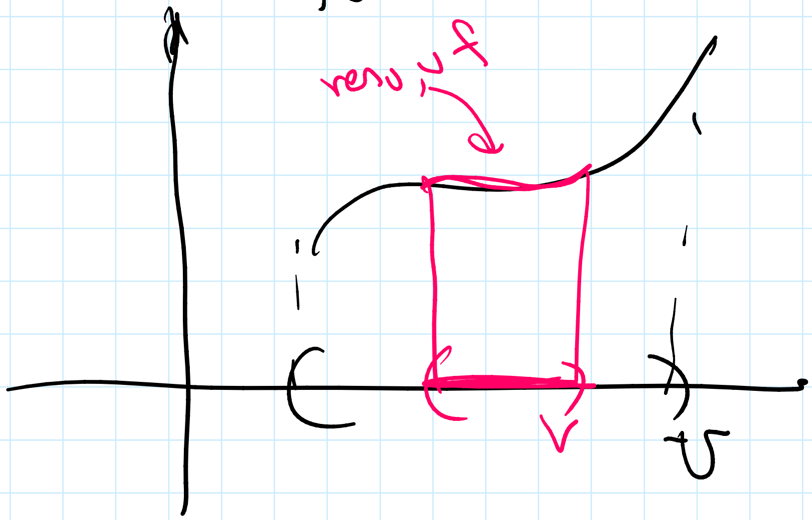
(2)



f = continuous on W

$res_{W,V}(f)$ = same function on V

$res_{V,U}(f)$ = same function on U

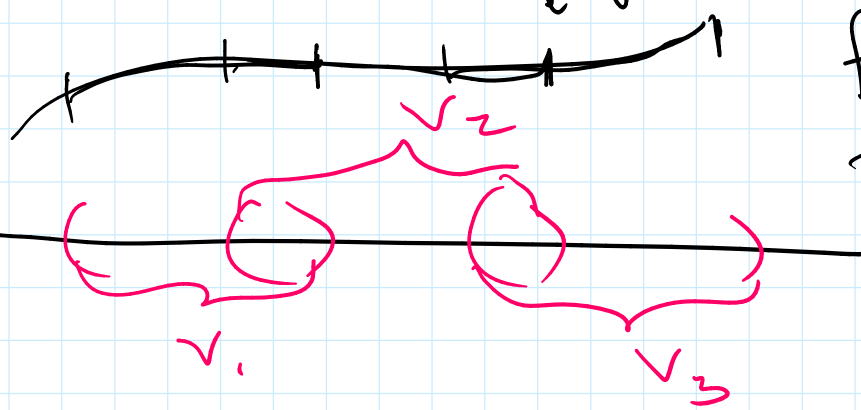


$res_{V,U}(res_{W,V}(f))$

$= res_{W,U}(f)$

(3) (Identity) $V = \bigcup_i V_i$

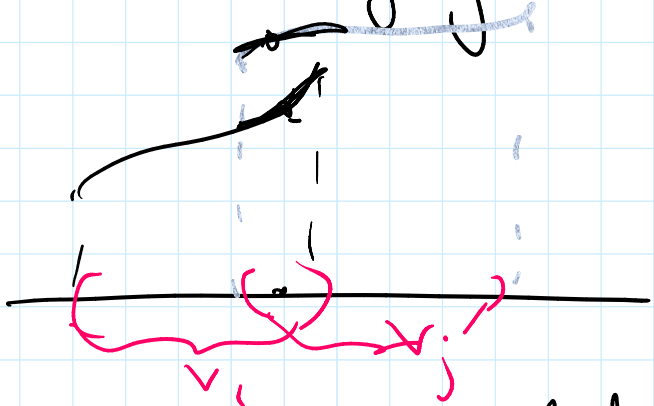
$res_{V,V_i} f = res_{V,V_i} g$ for all i



$f(x) = g(x)$ for all points x in V_i. Since union of V_i is the whole space V, we get $f(x) = g(x)$ for all x in $V \Rightarrow f = g$.

(4) (Gluing) We are given some

(4) (Gluing) We are given some functions f_i on V_i, V_i cover V. When do they glue to a function on V?



if $f_i = f_j$ on $V_i \cap V_j$ for all i, j then we can define $f(x) = f_i(x)$ if $x \in V_i$

If all f_i were continuous on V_i then f is continuous on V.

Also, similarly $X \subset \mathbb{R}^n$, we can define sheaves of differentiable functions etc.

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2 $\mathcal{O}_{\mathbb{P}^n}$ = structure sheaf / sheaf of functions on $\mathbb{P}^n$.

Use Zariski topology, cover $\mathbb{P}^n$ by charts $U_0, \ldots U_n$ $\quad U_i = \{x_i \neq 0\}$

U is open if $U \cap V_i$ is Zariski open in all U_i

U is open if $U \cap U_j$ is Zariski open in all U_i

$\mathcal{F}(U)$ = ring of algebraic functions on U

$\mathcal{F}(U_i)$ = polynomial function on $U_i = \mathbb{K}^n$

$\parallel$

$\{ f(\frac{x_0}{x_i}, \dots \frac{x_n}{x_i})$ polynomial $\}$ $\quad (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots; 1; \dots; \frac{x_n}{x_i})$

$\mathcal{F}(U_i \cap U_j)$ = algebraic functions on intersection. $\begin{pmatrix} x_i \neq 0 \\ x_j \neq 0 \end{pmatrix}$

$$\sum_k f_k\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) \cdot (x^j)^{-k}$$

= polynomial in $\frac{x_m}{x_i}$, $m \neq i, j$, Laurent polynomial in $\frac{x_j}{x_i}$,

$\mathcal{F}(U_i \cup U_j) = \{$ functions on U_i and U_j (f_i, f_j) which agree on $U_i \cap U_j \}$

Global functions $\mathcal{F}(\mathbb{P}^n) =$

$\{$collection of functions $f_i \in \mathcal{F}(U_i)$ (function on U_i) which agree on the intersection$\}$

Note: $\mathcal{F}(\mathbb{P}^n) = \mathbb{C}$ (constant function)

but $\mathcal{F}(U)$ interesting since

but $\mathcal{F}(U)$ interesting in general.

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③ Non-example: bounded functions on $\mathbb{R}$ (usual top.)

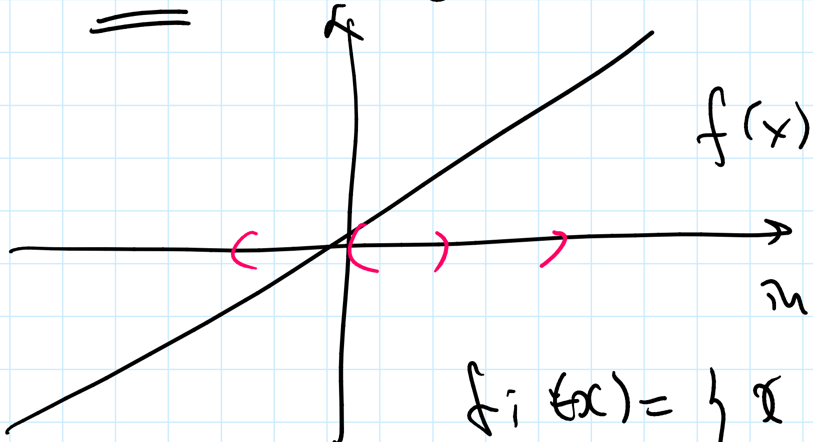
$\mathcal{F}(U) = \{$bounded continuous functions on $U\}$

$V \subset U \quad f$ bounded on U

$\Rightarrow f$ bounded on V.

$\Rightarrow res_{U,V}$ is defined, can check (1), (2), (3) $\Rightarrow$ presheaf.

But we do not have gluing!



$f(x) = x$ bounded on all intervals $(i, i+2) = U_i$

$f_i(x) = \{x$ for $x \in (i, i+2)\}$

agree on the intersection, but gluing is not bounded on $\mathbb{R}$,

④ Constant functions

$\mathcal{F}(U) = \{$constant functions on $U\}$

$F(U) = \{$ constant functions on $U \}$

Clearly, restriction works and F is a presheaf

Gluing does not work:

$f_1 = 1$ on $(1,2) = U_1$ $\qquad$ $U_1 \cap U_2 = \emptyset$

$f_2 = -1$ on $(3,4) = U_2$ $\qquad$ f_1 agrees with f_2 on the intersection

locally constant

Can glue f_1 and f_2 to a function $f = \begin{cases} 1 & \text{on } (1,2) \\ -1 & \text{on } (3,4) \end{cases}$ not constant!

not a sheaf!

__Easy fix__ $F(U) = \{$ locally constant functions on $U \}$

f is locally constant, if for every point there is neighborhood where f is constant.

__Rmk__ f is locally constant $\iff f' \equiv 0$

Rmk There is a general notion of

Rmk There is a general notion of "sheafification" which turns presheaves to sheaves

If we apply it to constant functions, we get locally constant ones.

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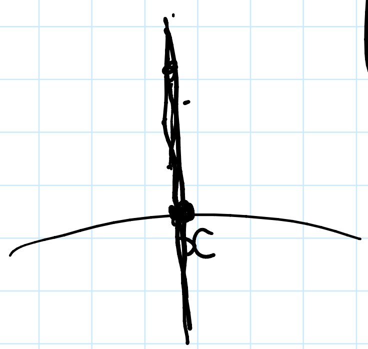
(5) "Skyscraper sheaf"

Fix a point $x \in X$

$$\mathcal{F}(U) = \begin{cases} \mathbb{C}, & \text{if } U \text{ contains } x \\ 0, & \text{otherwise.} \end{cases}$$



$V \subset U$ U contains x

- V contains x: res = id
- V does not contain x: $\mathcal{F}(V) = 0$, res = 0

This is a sheaf

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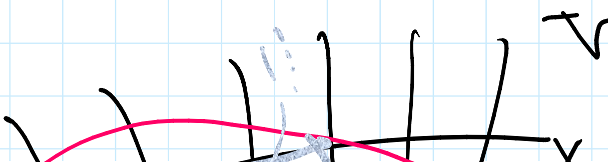
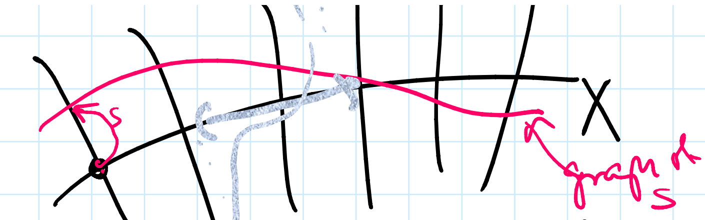
(6) $\mathcal{L}$ = line bundle on X

$$\mathcal{L}(U) = \left\{ \text{sections of } \mathcal{L} \text{ over } U \right\}$$



$U' \subset U$



$\pi : V \longrightarrow X \qquad s : U \to V$
$s : X \longrightarrow V$
such that $\pi(s(x)) = x$

$U' \subset U$

Can restrict a section from U to U' and get a section on U'

Can check that (1)-(4) are satisfied $\Rightarrow$ $\mathcal{L}(U)$ is a sheaf.

If U is small enough, $\mathcal{L}$ is trivial on $U \Rightarrow \mathcal{L}(U) = \{$functions on $U\}$

On $\mathbb{P}^n$, $\mathcal{L}(U_i) = \{$functions on $U_i\}$

<u>But</u>: gluing is interesting, depends on transition functions φ_{ij}.