Equivariant annular Khovanov homology

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4. Properties

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- For a cobordism S with d dots, its associated map has degree

$$-\chi(S) + 2d.$$

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- Extend $\mathcal{F}$ to surfaces with dots by interpreting a dot $\bullet$ as multiplication by $X \in A_0$.

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- The extensions (R, A), (R_α, A_α), and others were studied by Khovanov-Robert [KR20].

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$$R_0 = \mathbb{Z} \qquad R = \mathbb{Z}[E_1, E_2] \qquad R_\alpha = \mathbb{Z}[\alpha_0, \alpha_1]$$

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- (R, A) specializes to Lee's deformation by setting $E_1 = 0, E_2 = -1$ and to Bar-Natan's theory by setting $E_1 = 1, E_2 = 0$.

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- Expanding

$$(X - \alpha_0)(X - \alpha_1) = X^2 - (\alpha_0 + \alpha_1)X + \alpha_0\alpha_1,$$

we see that (R_α, A_α) is an extension of (R, A) via

$$E_1 \mapsto \alpha_0 + \alpha_1, \; E_2 \mapsto \alpha_0\alpha_1.$$

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  Hence the algebra structure on $A_{\alpha\mathcal{D}}$ decomposes,

$$A_{\alpha\mathcal{D}} = A_{\alpha\mathcal{D}} e_0 \times A_{\alpha\mathcal{D}} e_1.$$

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The link homology determined by $\mathcal{F}_{\alpha\mathcal{D}}$ is essentially Lee's theory.

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Proposition

Let L be an n component link with diagram D, and let $C_{\alpha\mathcal{D}}(D)$ denote the chain complex obtained by applying $\mathcal{F}_{\alpha\mathcal{D}}$ to the cube of resolutions. Then the homology of $C_{\alpha\mathcal{D}}(D)$ is a free $R_{\alpha\mathcal{D}}$-module of rank 2^n.

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Can be proven along the same lines as [BNM06, Weh08].

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- Annular homology is triply graded: in additional to homological and quantum grading, there is a third grading coming from winding around the annulus.

Annular homology

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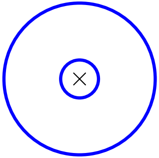

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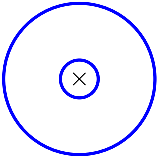

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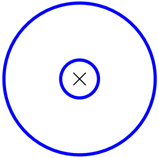

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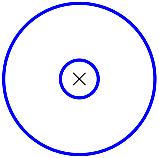

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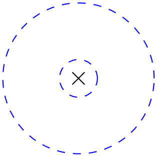

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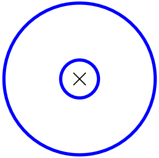

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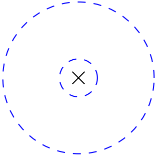

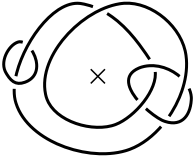

The annular TQFT $\mathcal{G}$

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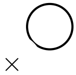
There are two types of circles in $\mathbb{A}$:



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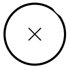

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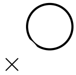

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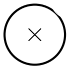

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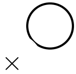

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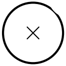

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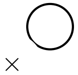

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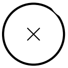

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So $\mathcal{G}(\mathscr{C})$ is a bigraded free abelian group via $(\text{deg}, \text{adeg})$.

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- In other words, $\mathcal{F}$ respects the adeg filtration, and $\mathcal{G}$ is the associated graded map.

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We distinguish the module assigned to trivial and essential circles. For a circle $C \subset \mathbb{A}$, write

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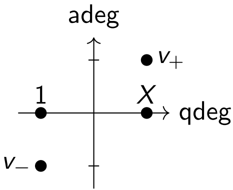

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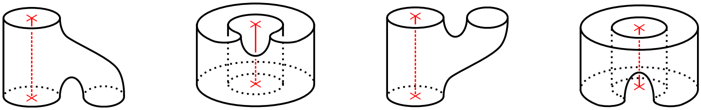

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Figure: Saddles involving essential circles

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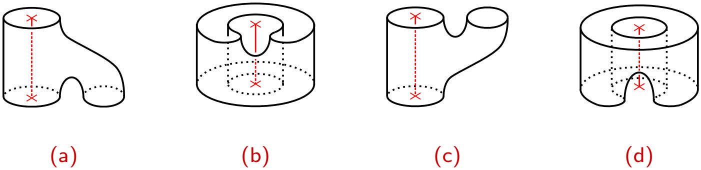

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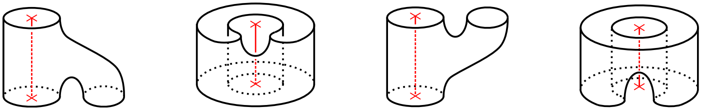

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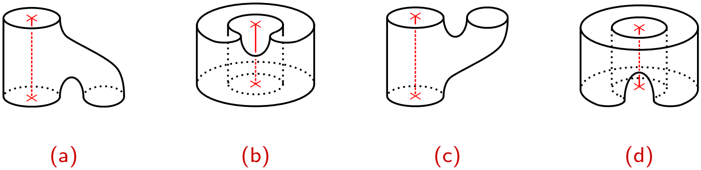

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We see that X acts trivially on an essential circle.

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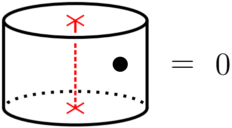

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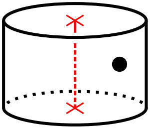 $= 0$

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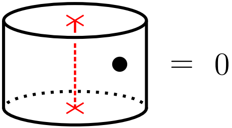

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This can't hold in the equivariant theories, since

$$X^2 - E_1X + E_2 = 0$$

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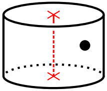

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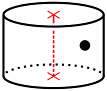

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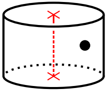

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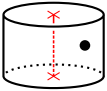

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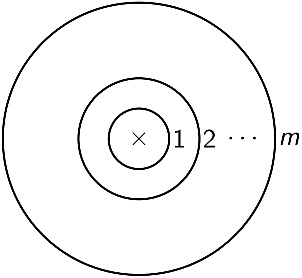

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- Say, the first essential circle is assigned $\{v_0, v_1\}$, the second is assigned $\{v_0', v_1'\}$, etc.

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Let $S \subset \mathbb{A} \times I$ be a cobordism from $\mathscr{C}_1$ to $\mathscr{C}_2$. Viewing S as a cobordism in $\mathbb{R}^2 \times I$, the map

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- Can set, say $\alpha_0 = 0$ and rename α_1 to $\alpha_1 = h$ to get an annular version of Bar-Natan homology.

The equivariant annular TQFT $\mathcal{G}_\alpha$

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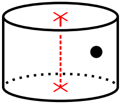
Key example: Let S denote



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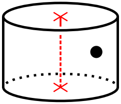

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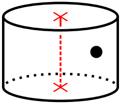

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$$\mathcal{F}_\alpha(S)(v_0)$$

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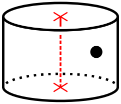

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$$\mathcal{F}_\alpha(S)(v_0) = X$$

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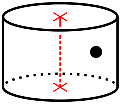

Recall $v_0 = 1$, $v_1 = X - \alpha_0$.

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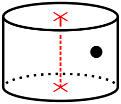

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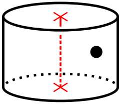

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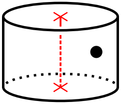

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Then the map $\mathcal{G}_\alpha(S)$ assigned to S is

$$\mathcal{G}_\alpha(S)(v_0) = \alpha_0 v_0$$
$$\mathcal{G}_\alpha(S)(v_1) = \alpha_1 v_1$$

More generally: let S denote the following cobordism

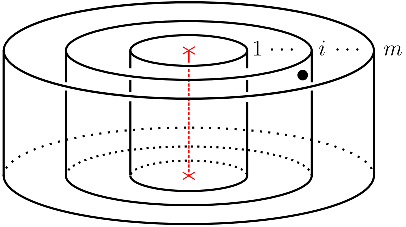

Figure: Product cobordism with the i-th component dotted

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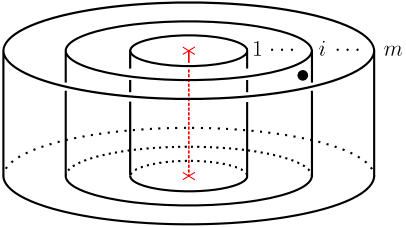

Figure: Product cobordism with the i-th component dotted

Then $\mathcal{G}_\alpha(S)$ is the identity on all tensor factors except the i-th, where it acts via

i odd	i even
$v_0 \mapsto \alpha_0 v_0$	$v'_0 \mapsto \alpha_1 v'_0$
$v_1 \mapsto \alpha_1 v_1$	$v'_1 \mapsto \alpha_0 v'_1$

The equivariant annular TQFT $\mathcal{G}_\alpha$

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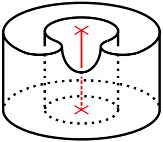

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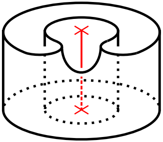

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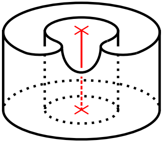

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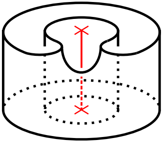

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Note that such a merge is always between consecutive essential circles.

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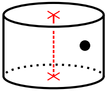

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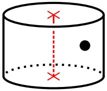

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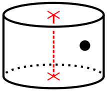

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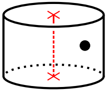

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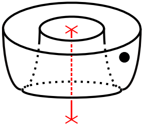

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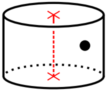

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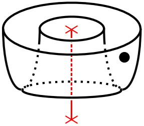

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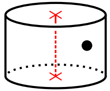

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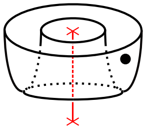

- Grigsby-Licata-Wehrli [GLW18] showed that the annular chain complex carries an action of $sl(2)$.
- This action also depends on parity of nesting.

Inverting the discriminant

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Theorem (A.)

Let $L \subset \mathbb{A} \times I$ be an n-component annular link with diagram D. Let $C^{\mathbb{A}}_{\alpha\mathcal{D}}(D)$ denote the chain complex obtained by applying $\mathcal{G}_{\alpha\mathcal{D}}$ to the cube of resolutions. Then the homology of $C^{\mathbb{A}}_{\alpha\mathcal{D}}(D)$ is a free $R_{\alpha\mathcal{D}}$-module of rank 2^n.

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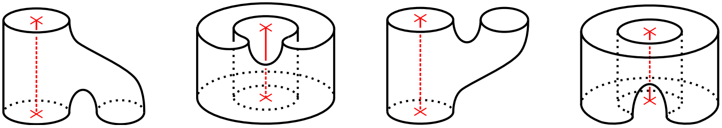
Proof: Consider the following elements of $A_{\alpha\mathcal{D}}$,

$$\overline{v}_0 := v_0 = 1, \qquad \overline{v}_1 := \frac{v_1}{\alpha_1 - \alpha_0} = \frac{X - \alpha_0}{\alpha_1 - \alpha_0},$$

$$\overline{v}'_0 := v'_0 = 1, \qquad \overline{v}'_1 := \frac{v'_1}{\alpha_0 - \alpha_1} = \frac{X - \alpha_1}{\alpha_0 - \alpha_1}.$$

Inverting the discriminant



(a) (b) (c) (d)

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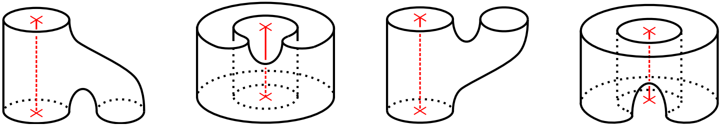

(a) (b) (c) (d)

(a)

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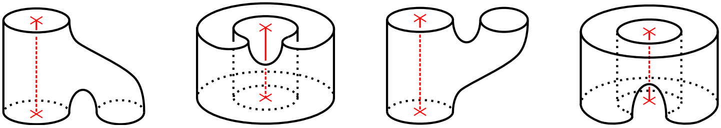
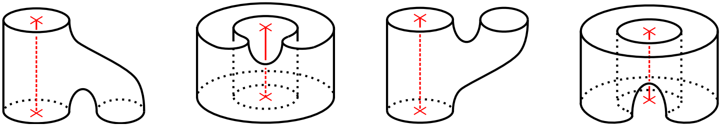


(a) (b) (c) (d)

(a)	(b)
$\overline{v}_0 \otimes e_0 \mapsto 0$	$\overline{v}_0 \otimes \overline{v}'_0 \mapsto 0$
$\overline{v}_1 \otimes e_0 \mapsto \overline{v}_1$	$\overline{v}_1 \otimes \overline{v}'_0 \mapsto e_0$
$\overline{v}_0 \otimes e_1 \mapsto \overline{v}_0$	$\overline{v}_0 \otimes \overline{v}'_1 \mapsto e_1$
$\overline{v}_1 \otimes e_1 \mapsto 0$	$\overline{v}_1 \otimes \overline{v}'_1 \mapsto 0$

- From (a), we see that $\overline{v}_0$ acts as e_1 and $\overline{v}_1$ acts as e_0.

Inverting the discriminant



(a) (b) (c) (d)

(a)

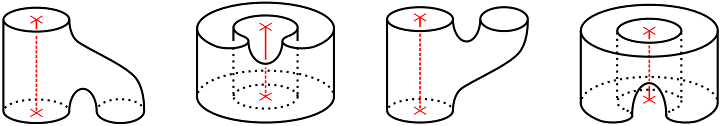
$$\overline{v}_0 \otimes e_0 \mapsto 0$$
$$\overline{v}_1 \otimes e_0 \mapsto \overline{v}_1$$
$$\overline{v}_0 \otimes e_1 \mapsto \overline{v}_0$$
$$\overline{v}_1 \otimes e_1 \mapsto 0$$

(b)

$$\overline{v}_0 \otimes \overline{v}'_0 \mapsto 0$$
$$\overline{v}_1 \otimes \overline{v}'_0 \mapsto e_0$$
$$\overline{v}_0 \otimes \overline{v}'_1 \mapsto e_1$$
$$\overline{v}_1 \otimes \overline{v}'_1 \mapsto 0$$

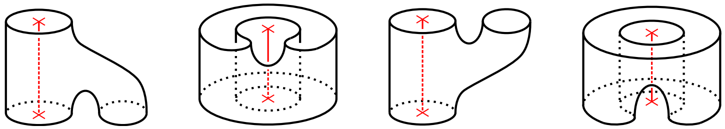
- From (a), we see that $\overline{v}_0$ acts as e_1 and $\overline{v}_1$ acts as e_0.
- From (b), $\overline{v}'_0$ acts as e_0 and $\overline{v}'_1$ acts as e_1.

Inverting the discriminant



(a) (b) (c) (d)

Inverting the discriminant



(a) (b) (c) (d)

(c)

$$\overline{v}_0 \mapsto (\alpha_0 - \alpha_1)\overline{v}_0 \otimes e_1$$
$$\overline{v}_1 \mapsto (\alpha_1 - \alpha_0)\overline{v}_1 \otimes e_0$$

(d)

$$e_0 \mapsto (\alpha_1 - \alpha_0)\overline{v}_1 \otimes \overline{v}'_0$$
$$e_1 \mapsto (\alpha_0 - \alpha_1)\overline{v}_0 \otimes \overline{v}'_1$$

Thank you!

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