M a 3-dim manifold

$x \in M.\ \xi_x \subseteq T_x M.$

ξ a 2-dim plane field (sub-bundle of TM)

α 1-form, locally $\xi = \mathrm{Ker}\alpha$

$\alpha \wedge d\alpha.\quad (\alpha\wedge d\alpha)_x = f(x)\cdot \omega_x.$

$$\text{If } \alpha\wedge d\alpha \begin{cases} = 0 \\ \text{positive}\cdot\text{volume form}: \alpha \text{ is} \\ \text{negative}\cdot\text{volume form} \end{cases} \begin{cases} \text{Foliation} \\ \text{Positive Contact Structure} \\ \text{Negative Contact Structure} \end{cases}$$

$f(x) \geq 0$, ≤ 0 Confoliation.

Thm. ξ is a foliation iff ξ is closed under Lie bracket.

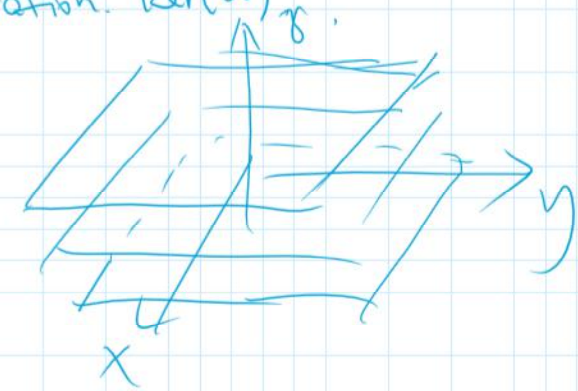
$u, v \in \xi_x.\ \alpha([u,v]) = \cancel{d_u(\alpha(v))} - \cancel{d_v(\alpha(u))} - d\alpha(u,v) = -d\alpha(u,v).$

$\alpha\wedge d\alpha \neq 0 \iff \exists w \notin \xi$, $u, v \in \xi.\ (\alpha\wedge d\alpha)(w,v,w) = \alpha(w)\, d\alpha(u,v)$

$\iff d\alpha(u,v) \neq 0 \iff [u,v] \notin \mathrm{Ker}\,\alpha \iff \xi$ not closed under $[\cdot,\cdot]$

Examples.

$\mathbb{R}^3$ $\alpha = dz.\ d\alpha = 0,\ \alpha\wedge d\alpha = 0.$ foliation. $\mathrm{Ker}(\alpha)_z$.



$\alpha = dz - y\,dx,\ d\alpha = -dy\wedge dx = dx\wedge dy$

$\alpha\wedge d\alpha = dx\wedge dy\wedge dz.$

positive Contact structure.

$\alpha = dz + y\,dx.$ negative " " "

$M = S^1 \times \Sigma$ ← 2-mfd. $\alpha = d\theta$ $\quad \{p\}\times\Sigma.$

S^1 parametrized by θ.

$S^3 \subseteq \mathbb{R}^4 = \mathbb{C}^2 \quad \alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2,\quad (r_1, \theta_1)$ polar coor. in $\mathbb{C}$

$\mathrm{Ker}\,\alpha = \langle r_2\partial_{r_1} - r_1\partial_{r_2},\ r_2^2\partial_{\theta_1} - r_1^2\partial_{\theta_2}\rangle$

$= \{v \in T_xS^3 \mid iv \in T_xS^3\},\quad \partial r_i \mapsto \frac{-\partial\theta_i}{r_i},\quad \partial\theta_i \mapsto r_i\partial r_i$

Lem. $\xi \overset{local\ coor.\ change}{\Longrightarrow} \mathrm{Ker}(dz - \alpha(x,y,z)dx)$.

$$\frac{\partial\alpha}{\partial y}\begin{cases} =0, & \text{foliation.} \\ >0, & \text{positive contact structure} \\ <0. & \text{negative} \end{cases}$$

$\mathrm{Ker}(dz)$.

$dz - y\,dx$.

$dz + y\,dx$.

Thm. ξ_t is 1-parameter family of contact structures on M, then there exists isotopy ψ: $M\times I \to M$, such that $(\psi_t)_*\xi_0 = \xi_t$.

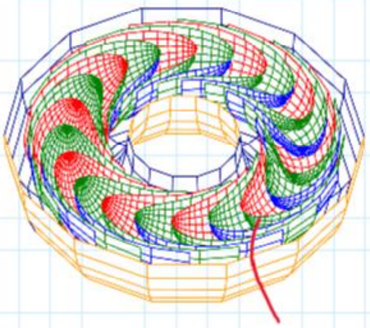
there exists isotopy $\psi: M \times I \to M$, such that $(\psi_t)_* \xi_0 = \xi_t$.

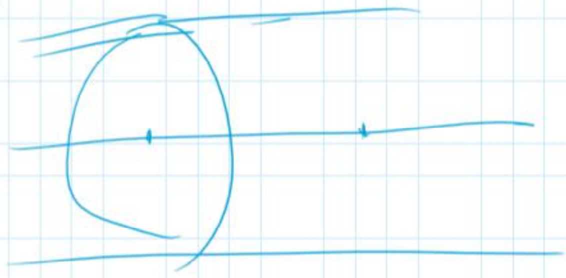
Reebless and Tight

Wednesday, October 14, 2020 11:04 AM

Reeb Foliation: Foliation as the level sets of $\sqrt{1-r^2}e^z$.

$T = \{r,\theta,z|r\le 1\}/(z\sim z+1)$





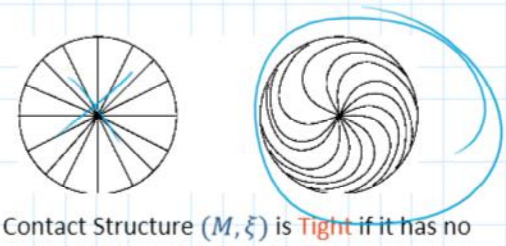
Foliation (M,ξ) is Reebless if the foliation has no reeb components.

Overtwisted Disk: $T = \{r,\theta,z|r\le\pi\}/\sim$, $D' = \{z=\epsilon r^2\}\subseteq T$

$\alpha = \cos r\, dz + r\sin r\, d\theta$ induces a foliation on D'

(The foliation given by $\xi\cap TD'\subseteq TD'$):

$\theta = \theta_0 - 2\epsilon\log\sin r$ away from $r=0$ (singular points)



Contact Structure (M,ξ) is Tight if it has no embedded overtwisted disks.

Thm. For a closed, oriented 3-mfd
Homotopy classes of plane fields $\Leftrightarrow$
Isotopy classes of overtwisted contact structures

closed

Thm. Reebless foliation or Tight positive contact structure ξ, embedded surface $\Sigma\subseteq M$ which is not a sphere. Then $|\langle e(\xi),[\Sigma]\rangle|\le-\chi(\Sigma)$.

$\in H^2(M)$ $\in H_2(M)$.

(global).

Pick vector field $v\in\xi$.

$\boxed{\text{0-locus of } v.}$

M': pushoff of M by v

$M\cap M'$: 1-mfd

$|[M\cap M']\cdot[\Sigma]| = |\langle e(\xi),[\Sigma]\rangle|$

Cor. Only finite many elements in $H^2(M,\mathbb{Z})$ can be the Euler class of some plain fields.

Taut and Weak symplectically semi-fillable:

Foliation ξ is taut if $\exists$ a closed curve intersects with all leaves transversally.

Contact structure ξ is WSSF if (M,ξ) is a component of (M',ξ') which is dominated by symplectic manifold (X,ω) ($\omega(v,w)>0$, (v,w) : oriented basis of ξ')

Contact structure ξ is WSSF if (M,ξ) is a component of (M',ξ') which is dominated by symplectic manifold (X,ω) $(\omega(v,w)>0$, (v,w) : oriented basis of $\xi')$

WSSF$\Longrightarrow$Tight, Taut$\Longrightarrow$Reebless. $\partial X = M'$. $d\alpha = \omega|_{\partial X}$.

Space of plane fields $\Longleftrightarrow$ Space of $\mathbb{P}^2$ valued functions on M.
(The normal vector of $\xi_x \subseteq T_x M$)

C^0-nbh.

θ. ξ' contact structure. foliation.

Special Foliation: $S^2 \times S^1$, $\zeta = \operatorname{Ker}(d\theta)$ given by $S^2 \times \{pt.\}$
Thm. Oriented C^2 foliation ξ on oriented 3-mfd M, other than $(S^2 \times S^1, \zeta)$, can be approximated by a positive/negative contact structure.

C^0

$\{\xi_{t_k}\}$. $k \to \infty$. $\xi_{t_k} \to \xi$.

Example: $\mathbb{T}^3$, $dz + t(\cos 2\pi nz\, dx + \sin 2\pi nz\, dy)$

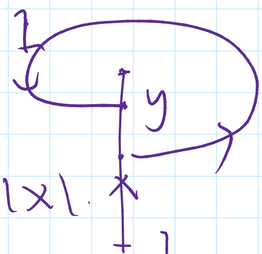
Why is $(S^2 \times S^1, \zeta)$ special?
Thm. If (M, ξ) contains 2-sphere $S \subseteq M$ and $T_x S = \xi_x$ for any $x \epsilon S$. Then $(M, \xi) \cong (S^2 \times S^1, \zeta)$
Any confoliation of $S^2 \times S^1$ is diffeomorphic to ζ in a C^0-nbh.

$\alpha \wedge d\alpha \geq 0$ (≤ 0).

Proof of the theorem.

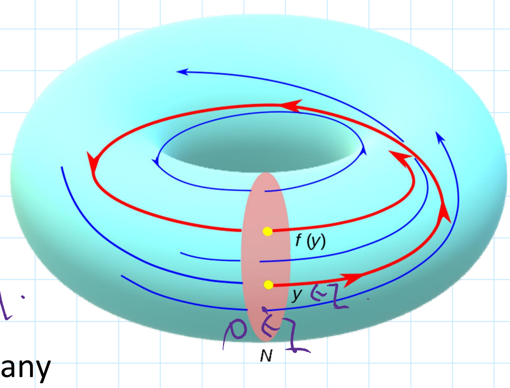
M.

Holonomy along a closed curve γ, which is tangent to ξ: the following map $\varphi: I \to I, x \longmapsto y$ ($I \times S^1$ embeds into M, S^1 into γ, I transverse to ξ)



Holonomy φ is:
Nontrivial, if $\varphi \neq id.$
Linearly nontrivial, if $\varphi'(0) \neq 1.$
Attracting/Repelling, if $|\varphi(x)| \lessgtr |x|.$
Sometimes Attracting/Repelling, if $\{x_i\}$ from both side $\to 0$. $|\varphi(x_i)| \lessgtr |x_i|$.

ξ



(a) We can C^0 perturb it into a foliation which has only finite many closed leaves.

Def. A minimal set: closed union of leaves which contains no closed union of leaves as a proper subset.

For a foliation after (a), M consists of:
Finite many closed leaves and some exceptional minimal sets (Minimal set which is neither closed leaf nor the entire mfd)
Or, M itself is a minimal set (ξ is minimal).

↳ linearly nontrivial holonomy.

If M is not minimal: it has linearly nontrivial holonomy (Sachsteder, 1965)
If M is minimal:
(a') Approximate ξ by a fiberation over S^1 (Tischler)
(a'') The fiber is not S^2! Approximate it by foliation with 2 closed leaves.

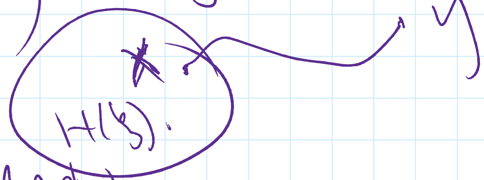
$\alpha \wedge d\alpha \geq 0$

(b) Thm. (M, ξ) is C^k- foliation, γ tangent to ξ, has linearly nontrivial holonomy. Then $\exists N \subset N' \subset M$, ξ can be C^k-deformed into a confoliation, which is positive contact in N, unchanged in N'^C, diffeomorphic to ξ in N^C.

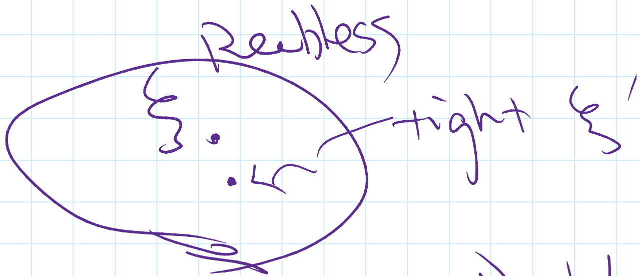
(c) If confoliation ξ has contact region $H(\xi)$, and any x connect to $H(\xi)$ by some path tangent to ξ.
Then ξ can be deformed into a contact structure.

$\{x \in M \mid (\alpha \wedge d\alpha)_x > 0\}$ $y \in M$, y

$$\begin{cases} \frac{\partial}{\partial t}\beta = *(\alpha \wedge df) \\ \beta(0) = \alpha. \end{cases} \quad f = *(\alpha \wedge d\beta + \beta \wedge d\alpha).$$



Thm. Reebless (Taut) foliation ξ approximated by a contact structure ξ', then ξ' is tight (WSSF).
The inverse is not always true!



$$\begin{tikzcd} \text{Foliation} \arrow[d, "\text{approx. by}"'] & \supset & \text{Reebless} \arrow[d] & \supset & \text{Taut} \arrow[d] \\ \text{Contact} & \supset & \text{Tight} & \supset & \text{WSSF} \end{tikzcd}$$

Contact structure $\supset$ Tight $\supset$ wssf cont. mfd. Sympl. mfd. (v, w) oriented basis ξ.

(M, ξ) weakly filled by (X, ω): $M = \partial X$, $\omega(v, w) > 0$. v transverse to TM.

(M, ξ) strongly filled by (X, ω): $M = \partial X$, $\exists$ Dialating vector field v near ∂X:

$\mathcal{L}_v \omega = \omega$, (Then $\alpha := \iota_v \omega$, $d\alpha = \omega$ and $\alpha \wedge d\alpha = \frac{1}{2}\iota_v(\omega \wedge \omega)$ is volume form.) of M.

And $\xi = \mathrm{Ker}\ \alpha$. $d\alpha = d\iota_v \omega = \mathcal{L}_v \omega = \omega$.

$\alpha(w)\omega(u, u')$

Strongly filled $\Longrightarrow$ Weakly filled.

$\alpha \wedge d\alpha(w, u, u') = \omega(v, w)\,\omega(u, u')$

Thm. (X, ω) weakly fills (M, ξ), then $\exists (X, \omega')$ strongly fills it.

$TM \to$ basis of ξ. ξ ξ $\Rightarrow M$ homology sphere.

* extra hypothesis.

Thm. (X_1, ω_1) (convex) strongly fills (M, ξ), (X_2, ω_2) strongly fills (M, ξ) with vector field points into X (Concave strongly). Then
$X = X_1 \cup_M X_2$ has symplectic form ω, $\omega|_{X_1} = \omega_1$, and away from a nbh of ∂X_2, $\omega|_{X_2} = c\omega_2$. $U_{\partial X_2}$

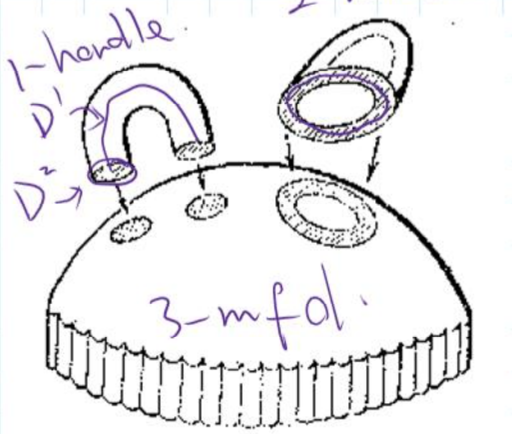
$c > 0$ const.

Attaching handles!

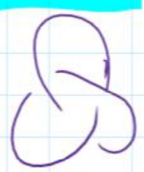
k-handle attached to n-mfd: A copy of $D^k \times D^{n-k}$ attached to ∂X along $\partial D^k \times D^{n-k}$.

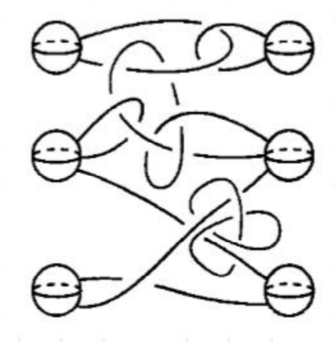
$\partial D' = 2$ pts

Handle decomposition of a (closed, connected) 4-mfd:
- A 0-handle. D^4. $\partial D^4 = S^4 \sim \mathbb{R}^3 \cup \{\infty\}$.
- Some 1-handles: Connect a pair of balls (In S^4) to each other.
- Some 2-handles: Attach along some thickened knots in ∂X_1, with framing.
- 3-handles and 4-handles are uniquely determined.



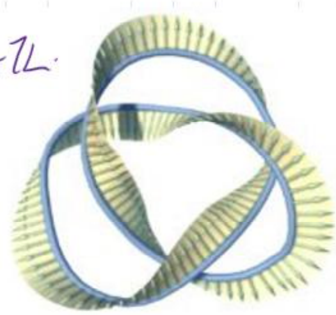
Kirby diagram:





$D^3 \times I$
$S^1 \times D^2$.

$n \in \mathbb{Z}$.



Thm (Elishaberg, Weinstein) (X, ω) with strong or weak convex boundary.
X' is derived by:
- Attaching 1-handles to X, or
- Attaching 2-handles: A knot K, $T_x K \subset \xi_x$. Normal bundle of ξ in TM is a ca~~x~~nonical framing (Contact framing). K: Lagrangian knot

- Attaching 1-handles to X, or
- Attaching 2-handles: A knot K, $T_x K \subset \xi_x$. Normal bundle of ξ in TM is a ca~~c~~nonical framing (Contact framing). K: ~~Lagrangian~~ knot.
  Attaching the 2-handle with framing 1 less than the contact framing.

Then the symplectic form extended to X', and the new boundary is still strong/weak convex!

Legendrian.

Thm (Elishaberg, Etnyre) Compact symplectic mfd (X, ω) with weak boundary can embed into a closed symplectic mfd (X', ω').

maybe additional 2-handles.