On: Categorified Invariants & the Braid Group

The Question: Can sutured annular Khovanov homology be used to solve the word / conjugacy problem in B_n?

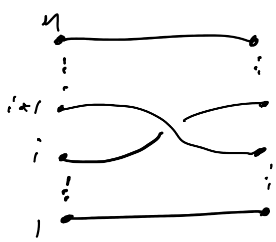
Prelims:

(Def): <u>n-strand Braid Group</u>:

$$B_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geq 2) \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & (|i-j| = 1) \end{array} \right\rangle$$

... where each σ_i is the Artin generator:

σ_i :=



- **Word Problem:** Let w & w' be words in the generators of B_n, & let $\sigma(w)$ denote the braid of w, then is $\sigma(w) = \sigma(w')$?
- **Conjugacy Problem:** $\sigma(w) \sim \sigma(w')$?

Answer:

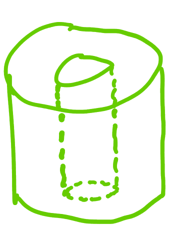
word problem? Yes!

conjugacy problem? Not Quite!

- Recall that Sutured Annular Khovanov homology associates to an oriented link $L \subset A \times I$ (A an annulus) a tri-graded **vector** space:

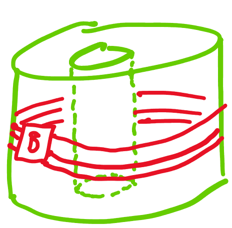
$$SKh(L) = \bigoplus_{i,j,k} SKh^i(L; j, k)$$

... $SKh(L)$ is an invariant of oriented isotopy **classes** of links $L \subset A \times I$. It is constructed as follows:



$A \times I$

... then an annular braid closure $\hat{\sigma}$ in $A \times I$ is:



... then we project $\hat{\sigma}$ onto the $A \times \{1/2\}$ "slice" of $A \times I$, like:

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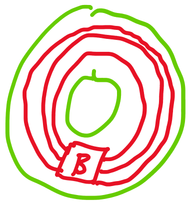
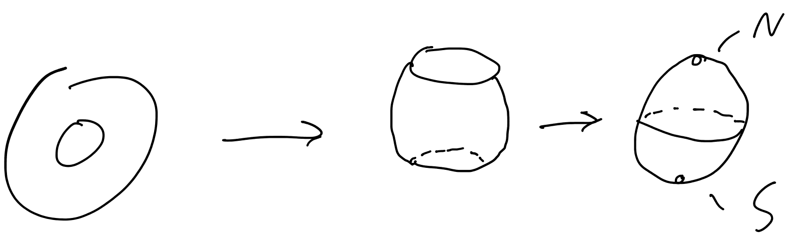


$A \times \{1/2\}$

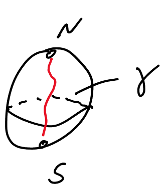
• **Note:** Conjugate braids have isotopic closures in $A \times I$.

... then we close up the inner & outer circles of $A \times \{1/2\}$, so we have our diagram D in $S^2 \setminus \{N, S\}$:

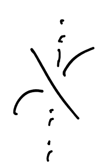


... if we forget about N, we have a diagram D in $S^2 \setminus \{S\} = \mathbb{R}^2$ and we can form the ordinary Khovanov complex of D, denoted $CKh_{*,*}$.

... to get SKh, choose an arc:



s.t. γ misses all crossings of $\hat{\sigma}$



(Def): k-grading: algebraic intersection # of the oriented resolutions of $\hat{\sigma}$ generating CKh and γ

• **Lemma:** d doesn't increase k-grading. (Roberts)

$\Rightarrow$ The k-grading gives us a filtration:

$$0 \subseteq \dots \subseteq \mathcal{F}_{n-1}(D) \subseteq \mathcal{F}_n(D) \subseteq \dots \subseteq CKh(D)$$

... $\mathcal{F}_n(D) :=$ subcomplex of $CKh(D)$ generated by resolutions w/ k-grading $\leq n$.

(Def): $\underline{\mathcal{F}_n(D;j)} = \mathcal{F}_n(D) \cap \bigoplus_i CKh^i(D;j)$

$\rightsquigarrow$ Sutured annular Khovanov homology groups:

$$SKh^i(L;j,k) = H^i\left(\frac{\mathcal{F}_k(D;j)}{\mathcal{F}_{k-1}(D;j)}\right)$$

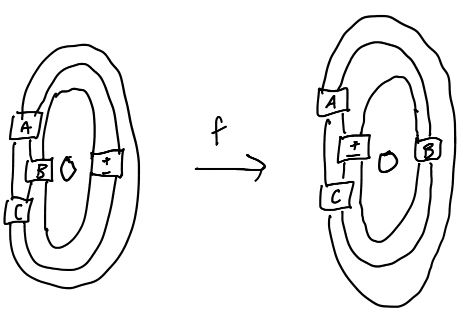
• **Thrm*** : $Skh(\hat{\sigma}) \cong Skh(\mathbb{1})$ implies $\sigma = \mathbb{1}$.

... So word problem solved, but what about the conjugacy problem?

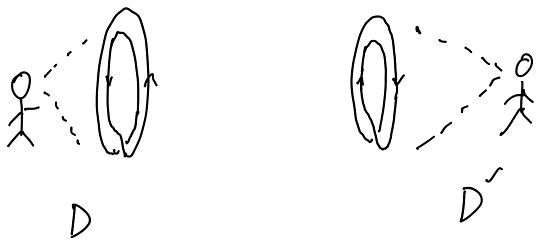
(Def): Reverse braid : For $\sigma \in B_n$, where $\sigma = \sigma(w)$, then the reverse of σ, denoted σ^r, is :

$$\sigma^r = \sigma(w^r)$$

(Def): Flype : Let σ & σ' be 3-braids then a flype is the following transformation:



... Note that, given projections D & D^r of closures $\hat{\sigma}$ & $\hat{\sigma}^r$ respectively, they look like:



Thm: Let $\sigma \in B_n$, then $SKh(\hat{\sigma}) \cong SKh(\hat{\sigma}^r)$.

proof. Given $\hat{\sigma}$ & $\hat{\sigma}^r$ and respective projections D and D^r onto $A \times \{1/2\}$, there is a bijective correspondance between the oriented resolutions of D & D^r, so $SKh(\hat{\sigma}) \cong SKh(\hat{\sigma}^r)$. ∎

- **Corollary:** $\exists$ infinitely many pairs $(\hat{\sigma}, \hat{\sigma}')$ s.t. $\hat{\sigma} \not\sim \hat{\sigma}'$ but $SKh(\hat{\sigma}) \cong SKh(\hat{\sigma}')$.

proof. Suppose $\hat{\sigma}$ & $\hat{\sigma}'$ are braids related by a flype, then $\hat{\sigma}'$ is isotopic to $\hat{\sigma}^r$ in $A \times I$, so $\hat{\sigma}'$ is a transverse mirror to $\hat{\sigma}$. Thus, by theorem, $SKh(\hat{\sigma}) \cong SKh(\hat{\sigma}')$.

... (Birman & Menasco) There are infinitely many distinct pairs of braids related by a flype s.t. the flype changes the conjugacy class. ∎

... but maybe ?

- **Open Questions:**

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(i) Does $Skh(\hat{\sigma}) \cong Skh(\hat{\sigma}')$ imply $\hat{\sigma} \sim \hat{\sigma}'$ or $\hat{\sigma} \sim \hat{\sigma}'^{r}$?

(ii) What if we also assume $\hat{\sigma}$ & $\hat{\sigma}'$ are alternating braids?

(iii) What if $Skh(\hat{\sigma}^{k}) \cong Skh(\hat{\sigma}'^{k})$ $\forall k \geq 0$?

... (ii) & (iii) combined? Link Floer homology is promising here, but ? for Skh.

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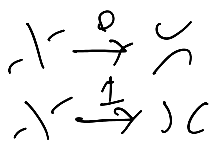
• Sketch of main theorem proof.



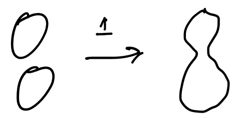
- Plamenevskaya's Invariant:

(Def): $\tilde{\psi}(L)$: Given link L with diagram given by braid closure $\hat{\sigma}$, take the oriented resolution obtained by taking the 0-resolution at positive crossings & the 1-resolution at negative crossings.

Let $\tilde{\psi}(L) = u_- \otimes u_- \otimes \cdots \otimes u_- \in V^{\otimes b}$.

• Lemma: $\tilde{\psi}(L)$ is a cycle.

Proof. d on $CKh(L_\alpha)$ is the sum of maps for all edges ξ with α as their initial end. Since $\tilde{\psi}(L) \in V^{\otimes n} = CKh(L_0)$, and when 0-res of +-crossing becomes a 1-res, the two circles merge:

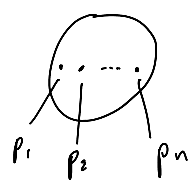


$\Rightarrow d_\xi$ is multiplication (recall: $m(u_- \otimes u_-) = 0$), so $d_\xi(\tilde{\psi}(L)) = 0$, & therefore $d(\tilde{\psi}(D)) = 0$. ∎

(Def): $\underline{\psi(L)}$: The homology class of $\tilde{\psi}(L)$.

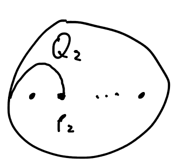
Thm: $\psi(L) \in Kh(L)$ is an invariant of transverse links $L \in (S^3, \xi_{std})$ up to sign.

(Def): $\underline{D_n}$:



(unit disk $\in \mathbb{C}$ with n points on $\mathbb{R}$-axis)

(Def): Admissible arc : $\gamma : I \longrightarrow D_n$ s.t. γ is a smooth embedding, γ transverse to ∂D, $\gamma(0) = -1 \in \mathbb{C}$, $\gamma(1) = p_i$, $i \in \{1, \ldots, n\}$, $\gamma(t) \in \mathrm{Int}(D_n) \setminus \{p_1, \ldots, p_n\}$ for $t \neq 0, 1$, and $\frac{d\gamma}{dt} \neq 0 \ \forall t$. So like:



Q_2 an admissible arc

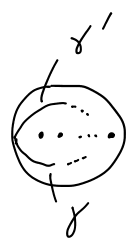
(Def): Pulled Tight : γ and γ' are pulled.

(Def): Pulled Tight : γ and γ' are pulled tight if $\gamma = \gamma'$ or γ intersects γ' transversally there are no empty bigons, like:

... $t_1, t_2, t_1', t_2' \in [0,1]$ then if $\gamma([t_1,t_2]) \cup \gamma'([t_1',t_2'])$ bounds a disk $A \subset D_n$, $\exists$ at least one $p_i \in A$.

• Lemma: Up to isotopy, we can always "pull tight" two admissible arcs.

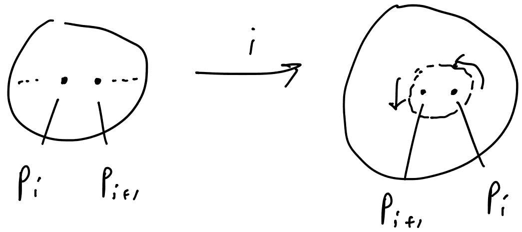
• (Def): Right : γ is right of γ' if, after they are pulled tight, orientation induced by tangent vectors $\frac{d\gamma}{dt}\Big|_{t=0}$ & $\frac{d\gamma'}{dt}\Big|_{t=0}$ agrees with the standard orientation on $D \subset \mathbb{C}$.



... So γ' starts above γ if γ right of γ'.

• Thm: $B_n \cong MCG(D_n)$

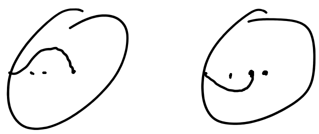
... identify each σ_i with $D_n \xrightarrow{i} D_n$ defined by :



i = id on D_n except for a small disc containing p_i & p_{i+1}, on this disc it's a ccw-rotation.

=> braids act on D_n from the right, so B_n acts on admissible arcs up to isotopy. $(\gamma)\sigma$

Merely convention, could choose negative Artin generators.



(Def): <u>Right Veering</u>: $\sigma \in B_n$ <u>right-veering</u> if, $\forall \gamma$, $(\gamma)\sigma$ is right of γ when pulled tight. (Left-veering same deal)

• <u>Lemma</u>: If $\sigma \in B_n$ is left & right veering, then $\sigma = \mathbb{1}$.

... follows from Alexander's lemma & fact that σ isotopic to a map that fixes all "nice" admissible arcs. ∎

• <u>Thm</u>: if σ isn't right-veering, then $\psi(\hat{\sigma}) = 0$.

• <u>Corollary</u>: If $\psi(\hat{\sigma}) \neq 0$ and $\psi(m(\hat{\sigma})) \neq 0$,

• **Corollary**: If $\psi(\hat{\sigma}) \neq 0$ and $\psi(m(\hat{\sigma})) \neq 0$, then $\sigma = \mathbb{1}$.

... filtration of CKh by k-grading yields a spectral sequence s.t.:

$$E_1 = SKh(L), \quad E_\infty \cong Kh(L)$$

& d_n shifts the triple grading (i,j,k) by $(1,0,-n)$.

proof. (of ★): Suppose $SKh(\hat{\sigma}) \cong SKh(\mathbb{1})$.

(Roberts): $\psi(\hat{\sigma})$ is the image of the bottom k-grading of $SKh(\hat{\sigma})$ under the spectral sequence mentioned above.

... A computation of the spectral sequence shows $SKh(\hat{\sigma})$ collapses immediately and $\psi(\hat{\sigma})$ survives.

• **Thrm** (Roberts): SKh symmetric under taking mirrors; $SKh^{i,j,k}(L) \cong SKh_{-i,-j,-k}(m(L))$, and the spectral sequence converging to $Kh(L)$ is filtered chain isomorphic to that induced on $SKh_{*,*,*}(m(L))$ by higher differentials on $SKh^{*,*,*}(m(L))$.

... symmetry implies the spectral sequence $SKh(m(\hat{\sigma}))$ to $Kh(m(\hat{\sigma}))$ collapses immediately & $\psi(m(\hat{\sigma})) \neq 0$

$\Rightarrow \sigma = \mathbb{1}$.

$\blacksquare$

- **Note:** Roberts defines "Khovanov Skein homology", which is an equivalent construction.

- **Remark:** All results hold for $\widehat{HFL}$.

- **Remark:** Speed of SKh?

(Garside - Thurston): $\exists$ an algorithmic solution to the word problem in B_n with complexity $O(|w|^2 n \log n)$.

... further improved to $O(|w|^2 n)$ (Birman - Ko - Lee)

Sources:

(Roberts): L. P. Roberts. "On Knot Floer homology in double branched covers"

(J. A. Baldwin & E. Grigsby): "Categorified invariants & the braid group"

(O. Plamenevskaya): "Transverse knots & Khovanov homology."