# Plamanevskaya's Invariant

Tuesday, December 1, 2020 3:16 AM

• Note: Kh coefficients in $\mathbb{Z}$.

(Def): Self-Linking #: Let $K \subset (M^3, \xi)$ be an oriented transverse link. Let $K$ bound seifert surface $\Sigma' \subset M$.

... let $\xi|_{\Sigma'} \to \Sigma'$ be a rk 2 vector bundle with non-vanishing section $s$, push $K$ in the $s$ direction to get copy $K^s$.

... $sl(K, [\Sigma']) :=$ algebraic $\cap$ # of $K^s$ and $\Sigma'$.

... If $L$ represented by $\hat{\sigma}$, $sl(L) = -b + n_+ - n_-$

• Thrm: $q$-grading of $\psi(L)$ is the self-linking # of $L$. In particular, $\psi(L) \in Kh^{0, sl(L)}(L)$.

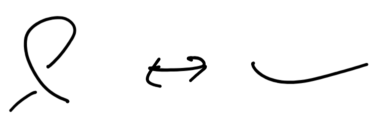
... recall $\xi_{..} := \ker(dz - ydx + xdy)$ for $S^3$;

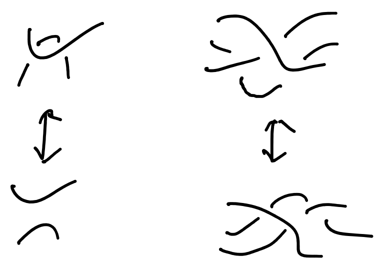
... recall $\xi_{std} := \ker(dz - y\,dx + x\,dy)$ for $S^3$; & any closed braid around the $z$-axis can be made transverse.

• **Thm** (Transverse Markov): Braids representing two transversely isotopic knots are related by a sequence of positive stabilizations, conjugations, & braid group identities.

(Recall)
... $b \in B_n$ related to $b' \in B_{n+1}$ by positive stabilization if $b' = b\sigma_n$.

(Recall)
... this yields the transverse Reidemeister moves on Braid diagrams:

+-stabilization $\longrightarrow$ R1 

Braid isotopies $\longrightarrow$ R2 & R3 

• **Lemma**: Let $D$ & $D'$ related by transverse Reidemeister moves, $\rho_i : CKh(D) \longrightarrow CKh(D')$ ($i \in \{1,2,3\}$) the induced Quasi-isomorphism, then

$$\rho_i(\tilde{\psi}(D)) = \pm \tilde{\psi}(D')$$

**Proof.** Check how $CKh(D')$ changes depending on $Ri$.

$\blacksquare$

$\Rightarrow$ $\psi(D) \in Kh(D)$ nice under transverse Reidemeister moves.

• **Thm**: $\psi(L) \in Kh(L)$ is an invariant of $L \in (S^3, \xi_{std})$ up to sign.

**Proof.**

**Need**: arbitrary transverse isotopy $L \to L'$ sends $\psi(L) \xrightarrow{*} \psi(L')$.

• **Note**: An arbitrary transverse isotopy $S$ can be smoothly modified into a composition of braid isotopies & $+$-stabilization. The modification leaves $L$ & $L'$ fixed, so cobordism $S$ admits a decomp:

$$S = S_1 \cup \dots \cup S_k$$

$\Rightarrow$ (Jacobsson) $f_S(\psi(L)) = f_{S_k} \circ \dots \circ f_{S_1}(\psi(L))$

$$= \pm \psi(L')$$

(If two diagrams of a link are related by $R_i$-moves, the induced iso is canonical up to sign.)

$\blacksquare$

## Properties of $\psi(L)$

- (Def) Transverse stabilization: $L$ given by braid closure $\hat{\sigma}$, a transverse stabilization of $L$ is an addition of an extra string and negative crossing.

- Thm: If $L$ is a transverse stabilization, then $\psi(L) = 0$. 1 (transverse stab $\Rightarrow \psi(L) = 0$)

- Thm: $L'$ obtained from $L$ by a 0-resolution of a positive crossing & $S$ the resulting cobordism ($f_S$ the induced map $Kh(L) \to Kh(L')$), then:

$$f_S(\psi(L)) = \pm \psi(L')$$

**2** (positive crossing resolution $\Rightarrow$ $\psi$ equal up to sign)

Proof. $S$ a composition of 1-handle attachment and R1:

1-handle attachment on $CKh(L_0)$ induces a comultiplication map,

R1 has the map $\rho_1$ (above), $\rho_1(u_-) = u_- \otimes u_-$,
$\rho_1(u_+) = u_+ \otimes u_- - u_- \otimes u_+$

$\Rightarrow \tilde{\psi}(L) \longmapsto \tilde{\psi}(L')$. ∎

• (Def): Quasipositive Braids: braid word is the product of conjugates $\omega \sigma_i \omega^{-1}$, $\omega \in B_n$.

• Thm: **3** If $L$ represented by quasi-positive braid, $\psi(L) \neq 0$.

• Note: Let $L$ be quasipositive braid w/ index $b$ with $n$-crossings, then $\psi(L)$ is a generator of $Kh^{0,n-b}$.

• **Thm:** $\psi$ Let $L$ represented by $\hat{\sigma}$ whose word contains a $\sigma_i^{-1}$ factor but no $\sigma_i$'s (only neg crossings between $(i-1)$th & $i$th string), then $\psi(L)=0$.

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• **Example:** $q>0$, $|p|\geq q$, $L :=$ transverse link of $(p,q)$-torus link type.

$\rightsquigarrow (p>0)$: If $L$ s.t. $sl(L)=pq-p-q$, then $\psi(L)$ is a generator of $Kh^{0,pq-p-q}\cong\mathbb{Z}$. Otherwise $\psi(L)=0$.

$\rightsquigarrow (p<0)$: $\psi(L)=0$.

... comes from fact that unique transverse positive $(p,q)$-torus link with $sl(L)=pq-p-q$ represented by $\sigma\in B_q$ with $p(q-1)$ crossings.

... If $L$ a positive braid of index $b$ with $n$ crossings, then $Kh^{0,n-b}\cong\mathbb{Z}$ of which

[illegible]

$n$ crossings, then $Kh^{0,n-b} \cong \mathbb{Z}$ of which $\psi(L)$ is a generator.

... $sl(L) < pq - p - q \Rightarrow L$ obtained by transverse stabilization $\Rightarrow \psi(L) = 0$ by above thm.