Annular Khovanov Lee Homology

J. ELISENDA GRIGSBY, ANTHONY M. LICATA, AND STEPHAN M. WEHRLI

1 Introduction

We are going to see a theorem on an invariant of Annular Khovanov Homology.

We will try to understand the Khovanov-Lee complex of an oriented link, L, in a thickened annulus, $A \times I$, has the structure of a $(Z \oplus Z)$–filtered complex whose filtered chain homotopy type is an invariant of the isotopy class of $L \subset (A \times I)$.

We will prove the theorem next time.

MAIN THEOREM

Let $L \subset (A \times I)$ be an annular link with wrapping number w, let o be an orientation on L, and let $t \in [0, 2]$.

1 $d_t(L, o)$ is an oriented annular link invariant.

2 $d_{1-t}(L, o) = d_{1+t}(L, o)$ for all $t \in [0, 1]$.

3 $d_0(L, o) = d_2(L, o) = s(L, o) - 1$.

4 Viewed as a function $[0, 2] \longrightarrow \mathbb{R}$, $d_t(L, o)$ is piecewise linear. Moreover, letting

$$m_t(L, o) := lim_{t \to 0^+} \frac{d_{t+\epsilon}(L, o) - d_t(L, o)}{\epsilon}$$

denote the (right-limit) slope at t, we have $m_t(L, o) \in \{-w, -w+2, \ldots, w-2, w\}$ for all $t \in [0, 2)$.

5 Suppose (L, o) and $(L', 0')$ are non-empty oriented annular links, and F is an oriented cobordism from (L, o) to (L', o') for which each component of F has a boundary component in L. Then if F has a_0 even-index annular critical points, a_1 odd-index annular critical points, and b_0 even-index non-annular critical points, then

$$d_t(L, o) - d_t(L', o') \leq (a_1 - a_0) - b_0(1 - t).$$

6 Suppose $(L, o), (L', o')$ and F are as in (5) above, and suppose that in addition each component of F has a boundary component in L'. Then

$$|d_t(L, o) - d_t(L', o')| \leq (a_1 - a_0) - b_0(1 - t).$$

2 Annular Link

Let A be an oriented annulus, and let $I = [0, 1]$.

We will use the identification, that $I \times I \cong D^2$.

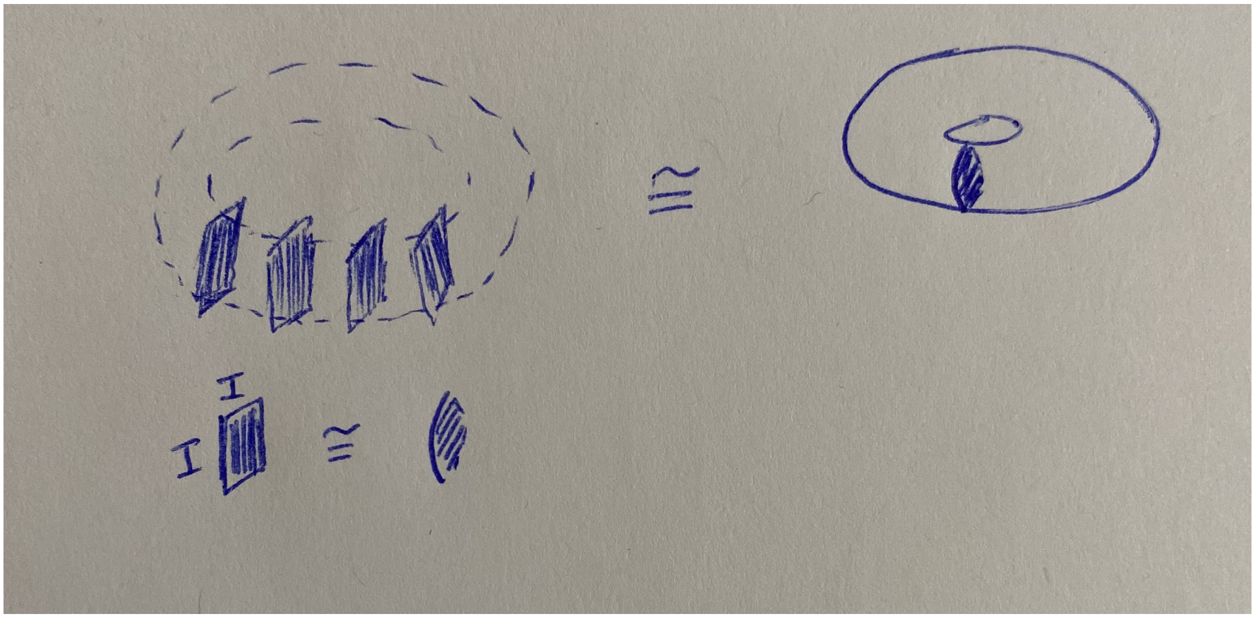
Consider a link $L \subset S^3$, and assume that L does not intersect the standard circle $(z - axis) \cup \{\infty\}$.

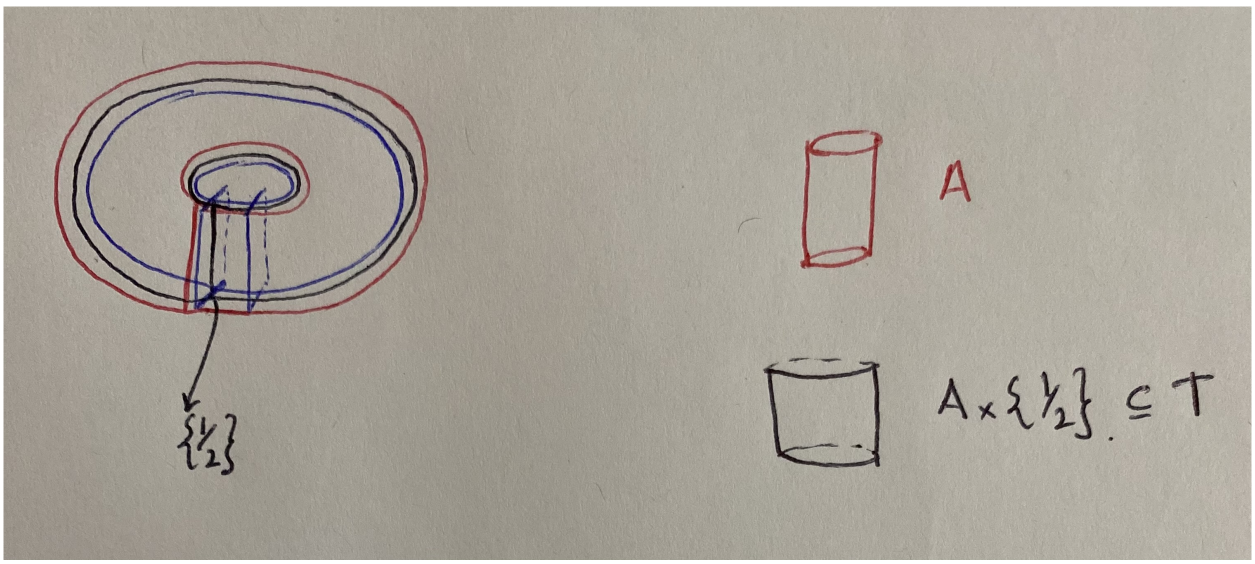
Now, consider the complement of this imbedded standard unknot in S^3. which is $S^3 - N(S^1)$ which is a solid torus T.

Now we know

$$T = S^1 \times D^2 \cong S^1 \times I \times I \cong A \times I.$$

We can think of this solid torus as a thickened annulus, as $L \subset A \times I$.





Now we consider the projection of L (any generic representation in the isotopy class of L) on $A \times \{\frac{1}{2}\}$ and denote the projection by $P(L)$.

$P(L)$ is the annular link diagram for L, in A.
We identify A with $S^2 \backslash \{X, O\}$ where X and O are two

punctures in S^2 each representing the two boundary components of the annular A.

Hence we can have the diagram $P(L) \subset S^2 \backslash \{X, O\}$.

Note that if we forget about $\{X\}$, then $P(L)$ could be thought of as a diagram in $S^2 \backslash \{O\} \cong \mathbb{R}^2$.

3 Annular Link Cobordism

Oriented Link Cobordism between oriented links (L, o) and $(L', o') \subset A \times I \subset S^3$; is a smooth, compact, properly embedded surface $F \subset S^3 \times I$ with

$$\partial F = \Big((L, o) \subset -S^3 \times \{0\}\Big) \sqcup \Big((L', o') \subset S^3 \times \{1\}\Big)$$

considered upto isotopy rel. boundary.

We call F, an annular cobordism if $F \cap (U \times I) = \phi$ where U denotes the standard embedded circle $(z{-}axis \cup \{\infty\})$.

An annular link cobordism F is said to be generic if the projection map

$$p : S^3 - U \longrightarrow I$$

restricted to F is Morse with distinct critical values.

An annular movie of a link is a smooth, 1-parameter family of curves, $D_t \subset S^2 \backslash \{O, X\}$; $t \in [0,1]$ called annular stills, satisfying

- For all but finitely many $t \in [0,1]$, D_t is a link diagram.[immersed curve equipped with over/under crossing]
- At each of the finitely many critical levels $t_1, t_2, \ldots t_k$, the diagrams undergoes a single elementary string interaction, namely: Birth, Deaths, Saddles and Reidemeister moves.

Since annular cobordisms are assumed to be compact hence we can consider all annular cobordisms to be imbedded in $(A \times I) \times I \subset (S^3 \backslash N(U)) \times I$.

Hence an annular movie may be viewed on $A \subset S^2 \backslash \{O, X\}$.

We end with a Lemma:

Lemma

Any annular link cobordism can be represented by an annular movie.

We will say that planar isotopy of $P(L)$ is annular if locally the move is supported in a disk away from $\{O, X\}$.

If locally it cannot be made disjoint then we call it non-annular.

By transversality we can say saddle (odd-index critical points) can be assumed to be annular.

There might exist planar isotopy which is non-annular.

Similarly for Reidemeister moves, it is possible that it might be non-annular move.

Fix an arc $\gamma \in S^2$ connecting X and O such that the arc misses every crossing of $P(L)$.

We call a circle C trivial if the algebraic intersection number of C with γ is even. On the other hand, it's non-trivial if it's odd.

Alternatively we can say, that a circle C is trivial if it bounds a disk in $S^2 \backslash \{O, X\}$.

A circle C is non-trivial if it fails to bounds a disk in $S^2 \backslash \{O, X\}$.

Even-index critical Points

A Birth(death) move is annular if there is an addition(deletion) of a trivial circle. If it is the addition(deletion) of a non-trivial circle then it is non-annular.

4 Annular Khovanov-Lee complex

For $P(L) \subset S^2 \backslash \{O, X\}$, we want to define a $\mathbb{Z} \oplus \mathbb{Z}-$filtered chain complex.

We give a partial order to $\mathbb{Z} \oplus \mathbb{Z}$ as $(a, b) \leq (a', b)$ if and only if $a \leq a'$ and $b \leq b'$.

Let C be a chain complex. We call C a $\mathbb{Z} \oplus \mathbb{Z}$ filtration if for each $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$, we have a subcomplex $F_{(a,b)} \subset C$, and whenever $(a, b) \leq (a', b')$ then $F_{(a,b)} \supset F_{(a',b')}$.

We form the bi-graded Khovanov-Lee complex

$$C = \bigoplus_{i,j \in \mathbb{Z}} C_{ij}$$

along with an endomorphism $\partial^{LEE} : C \longrightarrow C$, where $\partial^{LEE} = \partial + \Phi$, where both have degree 1 with respect to i(homological), and ∂ has degree 0 in $'j'$(quantum) and Φ has degree 4 in $'j'$.

As we already know,

$$\partial^2 = 0; \partial\Phi + \Phi\partial = 0; \Phi^2 = 0$$

We already know that the homology of the complex (C, ∂) OR $(C, \partial^{LEE}) = (C, \partial + \Phi))$. are invariants of L.

5 Remember X

Till now we have considered our link diagram in $S^2 \backslash \{0\} \cong \mathbb{R}^2$ and have our complex C (a finite dimensional bigraded complex.)

Remembering the data of X, we obtain a 3rd grading on the underlying vector space of the K-L complex.

We have a basis of C which corresponds $1-1$ with oriented Kauffman states of $P(L)$.
We identify a "v_+" marking on a component of a Kauffman state with a counterclockwise orientation of that component.

Define a "k" grading of a basis element c as the algebraic intersection number of the corresponding state with a fixed arc missing all crossings of $P(L)$.

v & w are the two different k grading.

$v_+ = 1$ $\quad$ $w_+ = 1$

$v_- = x$ $\quad$ $w_- = x$

<u>m on ∂</u>

$v_+ \otimes v_+ \longrightarrow 0$ *

$w_+ \otimes w_+ \longrightarrow w_+$

$v_+ \otimes w_+ \longrightarrow v_+$

$w_+ \otimes v_+ \longrightarrow v_+$

$v_- \otimes v_- \longrightarrow 0$

$v_- \otimes w_- \longrightarrow 0$

$w_- \otimes v_- \longrightarrow 0$

$w_- \otimes w_- \longrightarrow 0$

$v_+ \otimes v_- \longrightarrow w_-$

$v_+ \otimes w_- \longrightarrow 0$ *

$w_+ \otimes v_- \longrightarrow v_-$

$w_+ \otimes w_- \longrightarrow w_-$

$v_- \otimes w_+ \longrightarrow w_-$

$v_- \otimes w_+ \longrightarrow v_-$

$w_- \otimes v_+ \longrightarrow 0$ *

$w_- \otimes w_+ \longrightarrow w_-$

Δ on ∂

$v_+ \longrightarrow v_+ \otimes w_-$ *

$w_+ \longrightarrow w_- \otimes w_+ + w_+ \otimes w_- = v_- \otimes v_+ + v_+ \otimes v_-$

$v_- \longrightarrow v_- \otimes w_-$

$w_- \longrightarrow w_- \otimes w_-$

Lemma

$\partial^{LEE} = \partial_0 + \partial_- + \Phi_0 + \Phi_+$, where the (i,j,k) degrees are given by:

- $\deg(\partial_0) = (1, 0, 0))$
- $\deg(\partial_-) = (1, 0, -2))$
- $\deg(\Phi_0) = (1, 4, 0))$
- $\deg(\Phi_+) = (1, 4, 2))$

Corollary

The j and $j - 2k$ gradings on C endow (C, ∂^{LEE}) with the structure of a $\mathbb{Z} \oplus \mathbb{Z}-$ filtered complex.

Proof. From the previous lemma, ∂^{LEE} is non decreasing for j and $j - 2k$.
Consider the subcomplex

$$F_{(a,b)} = SPAN_{\mathbb{F}}\{x \in C | gr_{(j,j-2k)}(x) \geq (a, b).\}$$

Easy to check it is a $\mathbb{Z} \oplus \mathbb{Z}-$ filtration for j and $j - 2k$ gradings on C.

□

Let x be a $(j,k)-$ homogenous basis element of C, and let $t \in [0,2]$.

Define
$$j_t(x) = j(x) - t.k(x).$$

Corollary

For every t, j_t endows (C, ∂^{Lee}) with the structure of a (discrete, bounded) $\mathbb{R}$ filtered complex equipped with a finite filtered graded basis.

Proof. Same proof as before.
The j_t grading on C is still non-decreasing.
$$F_a = \{x \in C | j_t(x) \geq a\}.$$
This gives us the $\mathbb{Z} \oplus \mathbb{Z}-$filtered complex. □

6 Annular Rasmussen invariants of an annular link

We have $P(L) \subset S^2 \backslash \{O\} \cong \mathbb{R}^2$ and o is an orientation on L, then Lee describes $s_o \in C(P(L))$ whose homology class is non-zero.

Rasmussen used the $\mathbb{Z}-$ filtration induced by the $j-$ grading on C to define an invariant

$$s(K) \in 2\mathbb{Z},$$

where

$$s(K) = gr_j([s_o]) + 1 \in 2\mathbb{Z}.$$

Later Beliakova, Wehrli extended the result to

$$s(L, o).$$

If $\bar{o}$ is just the opposite orientation of o, then the result does not change.

Define:

$$d_t(L, o) = gr_{j_t}([s_o]) \in \mathbb{R}.$$

Lemma

Let $L \subset S^2 \backslash \{O, X\}$.
Consider $P(L) \subset S^2 \backslash \{O\}$ and $P'(L) \subset S^2 \backslash \{X\}$.
There exists an involution,

$$\theta : (C(P(L), \partial^{LEE})) \longrightarrow (C(P'(L), \partial^{LEE}))$$

inducing an isomorphism on homology.

It basically exchanges $v_\pm$ to $w_\pm$

Now we will understand the MAIN THEOREM

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