Notes: Khovanov homology, sutured Floer homology, and annular links by Grigsby and Wehrli

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1 Introduction

Heegaard Floer homology & Khovanov homology:

→ used in applications such as
- Knot concordance
- Dehn surgery
- contact geometry

→ different philosophies for theory's constructions
- Ozsváth - Szabó discovered algebraic relationship between homologies

$$\begin{bmatrix}\text{Khovanov-type} \\ \text{chain complex} \\ \text{assoc. to link}\end{bmatrix} \longleftrightarrow \begin{bmatrix}\text{Heegaard Floer-type} \\ \text{chain complex assoc.} \\ \text{to double-branched cover}\end{bmatrix}$$

For a link $L \subset S^3$ a spectral sequence whose

typical

$$E^2 \text{ term} = \widetilde{Kh}(\bar{L})$$

$$E^\infty \text{ term} = \widehat{HF}(\Sigma(S^3, L))$$

where $\widetilde{Kh}$ denotes reduced Khovanov homology

$\bar{L}$ denotes mirror of L

$\Sigma(A,B)$ denotes double-branched cover of A branched over B

$\widehat{HF}$ denotes Heegaard Floer homology

NOTE: Unless explicitly stated all Khovanov & Heegaard Floer homologies have coefficients in $\mathbb{Z}_2$

- Roberts building off of Plamenevskaya

Annular homology

→ relationship more useful & general than originally thought

Given a link L in the complement of a fixed unknot, $B \subset S^3$
∃ a spectral sequence from $Kh^*(L)$ to (a variant of) Knot Floer homology of $\tilde{B} \subset \Sigma(S^3, L)$
where $\tilde{B}$ is preimage of B in $\Sigma(S^3, L)$

Contact structures

→ established relationship

[(Kh) Plamenevskaya's tranverse invariant] ⟷ [(HF) Osváth-Szabó contact invariant]

- Baldwin & Plamenevskaya used extension of relationship to establish tightness of a number of non Stein-fillable contact structures.

- Grigsby & Wehrli (different direction)

Heegaard Floer homology for sutured mfolds (developed by Juhász)
simple variant of Khovanov homology categorifying the reduced, n-colored Jones polynomial *

— detects unknot whenever $n \geq 2$

Goal: Unify all these results

NOTE: Framework uses Gabai's sutured mfold theory
& Juhász's sutured Floer homology

Very nice! Can be shown to satisfy nice naturality props. WRT certain TQFT operations

Let F be oriented surface w/ $\partial F \neq \emptyset$,
$F \times I$ a product sutured mfold,
$T \subset F \times I$ a tangle (properly embedded 1-mfo
that is <u>BOTH</u>
- admissible : $T \cap (\partial F \times I) = \emptyset$
- balanced : $|T \cap (F \times \{0\})| = |T \cap (F \times \{1\})| = n \in \mathbb{Z}_{\geq 0}$

Then $\exists$ spectral sequence whose
E^2 term $= Kh^*(T)$

E^∞ term $= SFH(\Sigma(F \times I, T))$

NOTE: The appropriate version of Khovanov homology for balanced tangles in product sutured mfolds is similar to [1] w/ abelianized gradings where $F = A$ (an annulus) & $T = L$ is a 0-balanced tangle (link)

Main Theorems:

Thm 2.1: Let $L \subset A \times I$ be a link in the product sutured mfld $A \times I$. Then there is a spectral sequence whose E^2 term is $Kh^*(\overline{L})$ & whose E^∞ term is $SFH(\Sigma(A \times I, L))$.

NOTE: case when $F = D$ in [6]

NOTE: Reinterpretation (slight extension) of Rober[ts']
Main result

Thm 1.1: [6, Prop 5.20] Let $T \subset D \times I$ be an admissi[ble] balanced tangle. Then there is a spectral se[quence] whose E^2 term is, $Kh^*(\overline{T})$ & whose E^∞ ter[m] is $SFH(\Sigma(D \times I, T))$

NOTE: Roberts restricts to L intersecting a spanning disk of B in odd # of pt[s]
BUT Thm 1.1 makes NO restrictio[n]

Connection:
Let A be oriented annulus
$I = [0,1]$ oriented closed unit inter[val]
$L \subset A \times I$ link
where $A \times I$ identified as std sutured complement of standarly-imbedde[d] unknot $B \subset S^3$ w/ identification

$A \times I = \{(r, \theta, z) \mid r \in [1,2], \theta \in [0, 2\pi), z \in [\ldots$

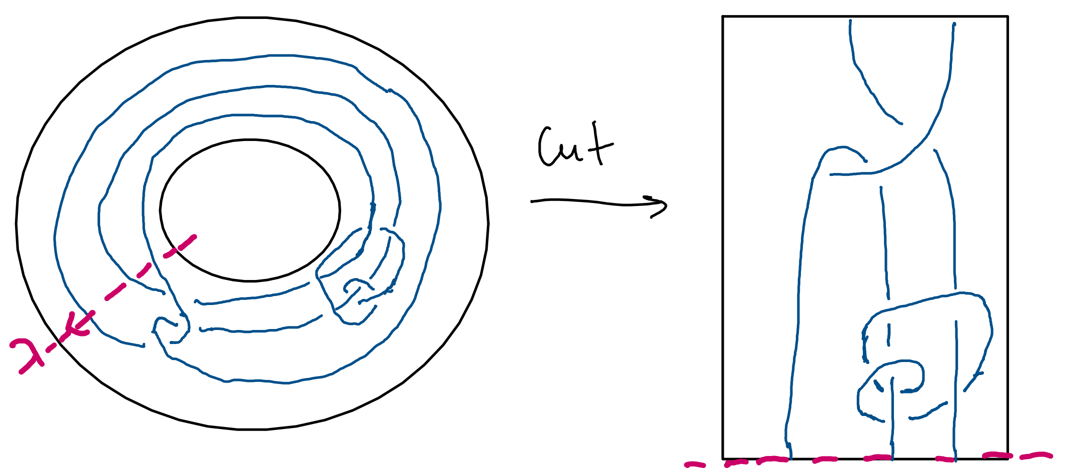
$A \times I = \{(r,\theta,z) : r \in [1,2], \theta \in [0,2\pi), z \in [$
$\subset \mathbb{R}^2 \cup \infty = S^3$

$B = \{(r,\theta,z) : r = 0\} \cup \infty \subset S^3$

Roberts constructs spectral sequence
$Kh^*(L)$ (for links in product mfld) $\longrightarrow$ Knot Floer homology of $\tilde{B} \subset \Sigma(S^3, L)$
($\tilde{B}$ preimage of $B \subset$

$\longrightarrow$ prop 2.24 shows that this (variant
Knot Floer homology of $\tilde{B}$ is the
sutured Floer homology of $\Sigma(A \times$

NOTE: nice relationship btwn spectral seq.s
Thm 1.1 & Thm 2.1
$\longrightarrow$ A link $L \subset A \times I$ can be cut al
vertical disk to form admissi
balanced tangle $T \subset D \times I$



Thm 3.1: Let $L \subset A \times I$ be an isotopy class represen
of an annular link admitting a projectio
$P(L)$ & let $\lambda \subset A$ be a properly imbedde

oriented arc representing a nontrivial element of $H_1(A, \partial A)$ s.t. λ intersect $\mathcal{P}(L)$ transversely. Let $T \subset D \times I$ be the balanced tangle in $D \times I$ obtained be decomposing $A \times I$ (def 2.8) along the sur $\lambda \times I$ endowed w/ the product orientat

Then the spectral sequence

$$Kh^*(\overline{T}) \longrightarrow SFH(\Sigma(D \times I, T))$$

is a direct summand of the spectral sequen

$$Kh^*(\overline{L}) \longrightarrow SFH(\Sigma(A \times I, L))$$

Moreover, the direct summand is trivial if $\exists\, L' \subset A \times I$ isotopic to L satisfying

$$|(\lambda \times I) \pitchfork L'| \lneq |(\lambda \times I) \pitchfork L|$$

NOTE: 1st example of "naturality" of relationsh btwn Kh & Heegaard Floer homology (under natural geometric operations) the spectral sequences behaves "as expected"

Interesting note: Given link $L \subset S^3$, any unknot $B \subset S^3 - N(L)$ endows the Khovanov chain complex associated to $L \subset B^3$ w/ $\mathbb{Z}$-filtration, via the identification

$$S^3 - N(B) \longleftrightarrow A \times I$$

The extra grading inducing $\mathbb{Z}$-filtrat
has representation-theoretic interpreti
$\longrightarrow$ Suppose $T \subseteq D \times I$ is an n-balan
tangle obtained by decomposing
$L \subset A \times I$ along $\lambda \times I$