Notes: Categorification of the Kauffaman bracket skein module of *I*-bundles over surfaces by Asaeda, Przytycki and Sikora

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1 Introduction

Khovanov constructed graded homology groups for links $L \subset \mathbb{R}^3$ & found polynomial Euler characteristic is Jones polynomial of L

→ paper uses Viro's (framed) normalization of Kh. groups

Khovanov used rep. of link by diagram in D^2 unique up to Reied. M.
BUT eventhough all I-bundles over surfaces allow analogous diagrammatic rep. doesn't extend straightforwardly to links

Goal: Define homology invariants of links in all orientable I-bundles M over surfaces $F \neq RP^2$

[Wikipedia]

I-bundles: fiber bundle whose fiber is an interval & base is a mfold
→ any kind of interval, open, closed, semi-open, semiclosed, open-bounded, compact & open rays can be the fiber

Ex: Only 2 I-bundles over S^1 are
- annulus (trivial nontwisted, oriented → $S^1 \times I$)
- Möbius band

Ex: Only 2 kinds of I-bundles when base mfold is any surface but Klein bottle

Klein bottle has 3 I-bundles
- trivial bundle $K \times I$
- 2 twisted bundles

NOTE: w/ Seifert fibered spaces, I-bundles fundamental elementary building blocks for description of 3-D spaces
→ elementary 3 mflds

Ex: line bundles are both I-bundles & vector bundles of rank 1

→ Analogou retract $N(M)$? skeleton

NOTE: for I-bundles more emphasis is placed on topological properties

def: A simple closed loop $\gamma \subset F$ is <u>bounding</u> if it bounds either a disk or a Möbius band in F.

Let $C(F)$ be the set of all unoriented, unbounding, simple closed curves of F up to homotopy

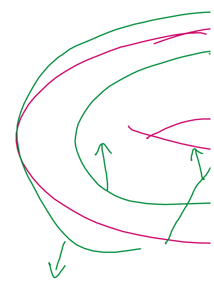
2 Skein modules of band links

Let M be an oriented I-bundle over surface F.

$$M = \begin{cases} \bullet\, F \times I \text{ for oriented } F \\ \bullet \text{ twisted } I\text{-bundle over an unorientable surface } F \\ \quad \rightarrow \text{assume } F \neq \mathbb{RP}^2 \end{cases}$$

def: A <u>band knot</u> K in M is either

- embedding of an annulus into M s.t. the projection of its core into F is a preserving orientation loop in F
- embedding of Möbius band into M s.t. the projection of its core into F is a reversing orientation curve in F

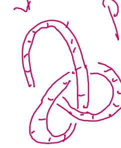


A disjoint union of band knot is called a <u>band link</u>

Each band link in M is represented by a diagram in F

NOTE: link diagrams considered w/ <u>blackboard framing</u>. An orientation reversing loop in F represents a Möbius band in M

flat in page



$\rightarrow$ All links are considered band links

<u>Notation</u>:

- $\mathcal{L}_b(M) = \{$all band links in an orientable I-bundle M over $F\}$

For given ring R w/ distinguished element $A^{\pm 1}$, skein module band links in M

- $S_b(M;R)$ = quotient of $R\mathcal{L}_b(M)$ by std Kauffman bracket skein rel:

$$\left.\begin{aligned} \diagup\!\!\diagdown &= A \asymp + A^{-1} \,)(\, \\ L \cup O &= -(A^2 + A^{-2})\, L \end{aligned}\right\}$$

NOTE: def of band link $L \subset M$ depends on I-bundle structure of M. The below theorem shows skein module $S_b(M;R)$ does not! (up to isomorphism of R-modules)

- $\mathcal{B}(F) = \{$all link diagrams in F w/no crossings & w/no trivial components$\}$ $\leftarrow$

- $\mathcal{B}_{nb}(F) = \left\{ \begin{array}{l} \text{all link diagrams in } F \text{ w/ no crossings} \\ \text{\& w/ no bounding components} \end{array} \right\}$

NOTE: $\emptyset \in \mathcal{B}(F)$ & $\emptyset \in \mathcal{B}_{nb}(F)$

Thm 2.1: $S_b(M;R)$ is a free R-module w/ a basis composed by (band) links represented by diagrams in $\mathcal{B}(F)$

It follows that $S_b(M;\mathbb{Z}[A^{\pm 1}]) = S(M;\mathbb{Z}[A^{\pm}])$ as R-modules even though no explicit isomorphism between these modules for unorientable F.

$\exists$ natural isomorphism between $S_b(M;R)$ & $S(M;R)$ for any ring R containing $\sqrt{-A}$

isomorphism $\lambda : S_b(M;\mathbb{Z}[A^{\pm 1}]) \longrightarrow S(M;\mathbb{Z}[A^{\pm 1}])$

where $L = K_1 \cup \dots \cup K_n \longmapsto (-A)^{3K(L)/2} K_1' \cup \dots \cup K_n'$

where $K_i' = \begin{cases} K_i & \text{if } K_i \text{ is annulus} \\ K_i + \text{neg. half twist} & \text{otherwise} \end{cases}$

$K(L) = \#$ of Möbius bands in components K_i of L

3 Chain Groups

Let D be link diagram in F

def: A Kauffman state of D is an assignment of $+1/-1$ marker to each crossing of D

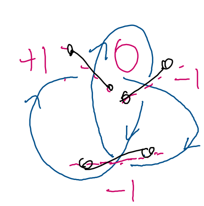
def: An enhanced (Kauffman) state of D is a Kauffman state w/ additional assignment of $+/-$ to each closed loop obtained by smoothing the crossings of D w/ convention

such closed loops called circles

Denote set of enhanced states of D by $S(D)$
For any $S \in S(D)$

- $I(S) = \#\{\text{pos. markers}\} - \#\{\text{neg. markers}\}$
- $J(S) = I(S) + 2\tau(S)$

where $\tau(S) = \#\{\text{pos. trivial circles}\} - \#\{\text{neg. trivial circles}\}$

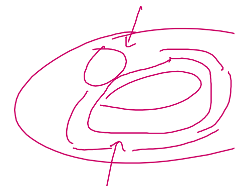


$I(S) = 1 - 2 = -$

? $J(S) = -1 + 2\tau$

$= -1$?

A circle is *trivial* if it bound a disk in F.
A circle is *bounding* if it bounds either a disk or a Möbius band in F.



$\in \mathcal{C}(F)$

Let $\mathcal{C}(F) = \left\{ \begin{array}{l} \text{all unoriented, unbounding, simple closed curves} \\ \text{in } F \text{ (up to homotopy)} \end{array} \right\}$

If unbounding components of an enhanced state S are $\gamma_1, \dots, \gamma_n$ (some may be parallel to eachother & equal in $\mathcal{C}(F)$)
& if these closed curves are marked by $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ then

$$\psi(S) = \sum_i \varepsilon_i \gamma_i \in \mathbb{Z}\mathcal{C}(F)$$

meant to be $I(S)$ $J(S)$

Let $S_{ijs}(D) = \{\text{enhanced states } S \text{ of } D : i(S) = i,\ j(S) = j,\ \psi(S) = s\}$

$C_{ijs}(D)$ = free abelian group spanned by enhanced states in $S_{ijs}(D)$

NOTE: $C_{**s}(D) = 0$ if $s = \sum_{i=1}^{n} \varepsilon_i \gamma_i$ & curves γ_i, $i = 1, \dots, n$ cannot be placed disjointly in F

If curves γ_i, $i = 1, \dots, n$ can be placed disjointly in F then s reps. a unique element of $B(F)$

So index $s \in \mathbb{Z}\mathcal{C}(F) \rightsquigarrow b \in B(F)$

4 Differentials

defns in Kh chain groups of link in $\mathbb{R}^3$ depend only on the choice of its diagram D.
BUT defn of differentials require an ordering of crossings of D.
→ also appears in this construction

Assume crossings of a link diagram D in F are ordered
Let the differential

$$d_{ijs} : C_{ijs}(D) \longrightarrow C_{i-2,j,s}(D)$$

$C_{1,2,3} \to C_{-1,2,3}$

for enhanced states $S \in C_{ijs}(D)$

$$d_{ijs}(S) = \sum_v (-1)^{t(S,v)} d_v(S) \quad \text{for crossings } v \in D$$

"partial derivative in direction of v"

$$d_v(S) = \sum_{S' \in S(D)} [S : S']_v \, S'$$

$[S:S']_v$: incidence number

$[S:S']_v = 1$
If all conditions are satisfied:

a) crossing v marked by $+ \in S$, $- \in S'$

b) S & S' assign same markers to all the other crossings

c) labels of common circles in S & S' are unchanged

d) $J(S) = J(S')$, $\Psi(S) = \Psi(S')$

$[S:S']_v = 0$ otherwise

$t(S,v) = $ # of neg. markers assigned to crossings in S bigger than v

thrown in (-) signs for asthetic pleasure

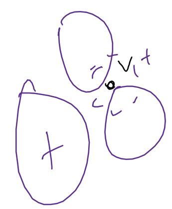
For fixed D, i, j, s we write

$$\hat{d}_v(S) = (-1)^{t(S,v)} d_v(S)$$

$$\therefore \quad d_{ijs}(S) = \sum_{\substack{\text{crossings} \\ v \in D}} \hat{d}_v(S)$$

for any $S \in S_{ijs}(D)$

S



$\tau(S) =$

If $[S:S']_v = 1$ then $\tau(S) = \tau(S') + 1$