Contact Structures and Foliations

Wednesday, October 14, 2020 10:39 AM

M a 3-dim manifold

$x \in M.\ \xi_x \subseteq T_x M.$

ξ a 2-dim plane field (sub-bundle of TM)

α 1-form, locally $\xi = \mathrm{Ker}\alpha$

$\alpha \wedge d\alpha.\quad (\alpha\wedge d\alpha)_x = f(x)\cdot \omega_x.$

If $\alpha \wedge d\alpha \begin{cases} = 0 \\ \text{positive} \cdot \text{volume form} \\ \text{negative} \cdot \text{volume form} \end{cases}$: α is $\begin{cases} \text{Foliation} \\ \text{Positive Contact Structure} \\ \text{Negative Contact Structure} \end{cases}$

$f(x) \geq 0$, ≤ 0 Confoliation.

Thm. ξ is a foliation iff ξ is closed under Lie bracket.

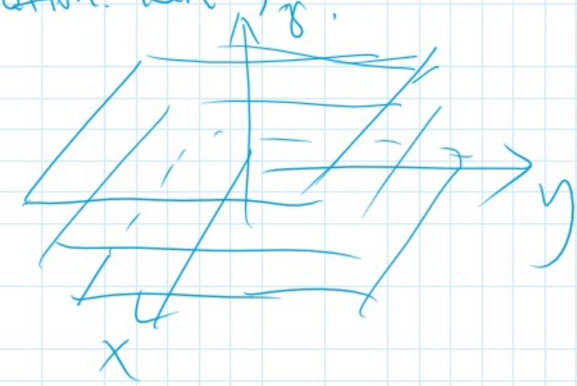
$u, v \in \xi_x,\ \alpha([u,v]) = \cancel{d_u(\alpha(v))} - \cancel{d_v(\alpha(u))} - d\alpha(u,v) = -d\alpha(u,v).$

$\alpha \wedge d\alpha \neq 0 \Leftrightarrow \exists\, w \notin \xi,\ u, v \in \xi.\ (\alpha\wedge d\alpha)(w,v,w) = \alpha(w)\, d\alpha(u,v).$

$\Leftrightarrow d\alpha(u,v) \neq 0 \Leftrightarrow [u,v] \notin \mathrm{Ker}\,\alpha \Leftrightarrow \xi$ not closed under $[\cdot,\cdot]$.

Examples.

$\mathbb{R}^3\ \alpha = dz.\ d\alpha = 0,\ \alpha\wedge d\alpha = 0.$ foliation. $\mathrm{Ker}(\alpha)$



$\alpha = dz - y\,dx,\ d\alpha = -dy\wedge dx = dx \wedge dy$

$\alpha \wedge d\alpha = dx\wedge dy \wedge dz.$ positive contact structure.

$\alpha = dz + y\,dx.$ negative " " "

$M = S^1 \times \Sigma$ ← 2-mfd. $\alpha = d\theta.$ $\{p+\} \times \Sigma.$ S^1

S^1 parametrized by θ.

$S^3 \subseteq \mathbb{R}^4 = \mathbb{C}^2\quad \alpha = r_1^2\, d\theta_1 + r_2^2\, d\theta_2,\quad (r_1, \theta_1)$ polar coor. in $\mathbb{C}$

$\mathrm{Ker}\,\alpha = \langle r_2 \partial_{r_1} - r_1 \partial_{r_2},\ r_2^2 \partial_{\theta_1} - r_1^2 \partial_{\theta_2} \rangle$

$= \{ v \in T_x S^3 \mid iv \in T_x S^3 \},\quad \partial_{r_i} \mapsto \frac{-\partial_{\theta_i}}{r_i},\quad \partial_{\theta_i} \mapsto r_i \partial_{r_i}$

Lem. $\xi \overset{local\ coor.\ change}{\Rightarrow} \mathrm{Ker}(dz - \alpha(x,y,z)dx).$

$\frac{\partial \alpha}{\partial y} \begin{cases} = 0, & \text{foliation.} \\ > 0, & \text{positive} \\ < 0. & \text{negative} \end{cases}$ contact structure.

$\mathrm{Ker}(dz).$

$dz - y\,dx.$

$dz + y\,dx.$

Thm. ξ_t is 1-parameter family of contact structures on M, then there exists isotopy ψ: $M \times I \to M$, such that $(\psi_t)_* \xi_0 = \xi_t$.

there exists isotopy $\psi\colon M \times I \to M$, such that $(\psi_t)_* \xi_0 = \xi_t$.

Reebless and Tight

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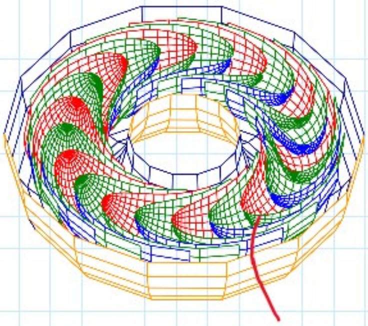
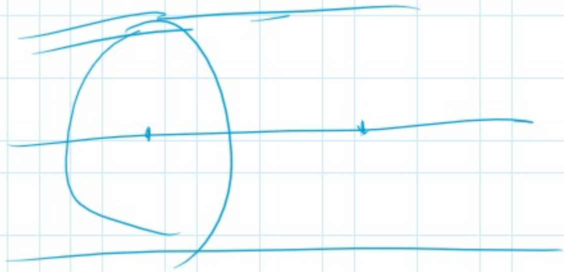
Reeb Foliation: Foliation as the level sets of $\sqrt{1-r^2}e^z$.

$T = \{r,\theta,z|r\le 1\}/(z\sim z+1)$





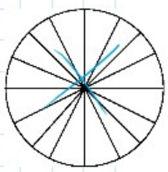
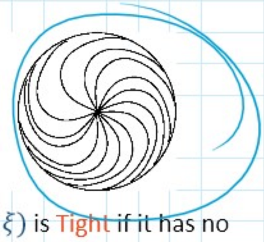
Foliation (M,ξ) is Reebless if the foliation has no reeb components.

Overtwisted Disk: $T = \{r,\theta,z|r\le\pi\}/\sim$, $D' = \{z=\epsilon r^2\}\subseteq T$

$\alpha = \cos r\, dz + r\sin r\, d\theta$ induces a foliation on D'

(The foliation given by $\xi\cap TD'\subseteq TD'$):

$\theta=\theta_0$-$2\epsilon\log\sin r$ away from $r=0$ (singular points)

Contact Structure (M,ξ) is Tight if it has no embedded overtwisted disks.

Thm. For a closed, oriented 3-mfd

Homotopy classes of plane fields$\Leftrightarrow$

Isotopy classes of overtwisted contact structures

(global).

Pick vector field $v\in\xi$.

$\boxed{\text{0-locus of } v.}$

closed

Thm. Reebless foliation or Tight positive contact structure ξ, embedded surface $\Sigma\subseteq M$ which is not a sphere. Then $|\langle e(\xi),[\Sigma]\rangle|\le-\chi(\Sigma)$.

$\in H^2(M)$ $\in H_2(M)$.

M': pushoff of M by v

$M\cap M'$: 1-mfd.

$|[M\cap M']\cdot[\Sigma]| = |\langle e(\xi),[\Sigma]\rangle|$

Cor. Only finite many elements in $H^2(M,\mathbb{Z})$ can be the Euler class of some plain fields.

Taut and Weak symplectically semi-fillable:

Foliation ξ is taut if $\exists$ a closed curve intersects with all leaves transversally.

Contact structure ξ is WSSF if (M,ξ) is a component of (M',ξ') which is dominated by symplectic manifold (X,ω) ($\omega(v,w)>0$, (v,w) : oriented basis of ξ')

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WSSF⟹Tight, Taut⟹Reebless. $\partial X = M'$. $d\alpha = \omega|_{\partial X}$.

Foliations approximated by Contact Structures

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Space of plane fields $\Leftrightarrow$ Space of $\mathbb{P}^2$ valued functions on M.
(The normal vector of $\xi_x \subseteq T_x M$)

θ.

$\{\xi_{t_k}\}$.
$k \to \infty$.
$\xi_{t_k} \to \xi$.

Special Foliation: $S^2 \times S^1$, $\zeta = \mathrm{Ker}(d\theta)$ given by $S^2 \times \{pt.\}$
Thm. Oriented C^2 foliation ξ on oriented 3-mfd M, other than $(S^2 \times S^1, \zeta)$, can be approximated by a positive/negative contact structure.

C^0

Example: $\mathbb{T}^3$, $dz + t(\cos 2\pi nz\, dx + \sin 2\pi nz\, dy)$

Why is $(S^2 \times S^1, \zeta)$ special?
Thm. If (M, ξ) contains 2-sphere $S \subseteq M$ and $T_x S = \xi_x$ for any $x \epsilon S$. Then $(M, \xi) \cong (S^2 \times S^1, \zeta)$

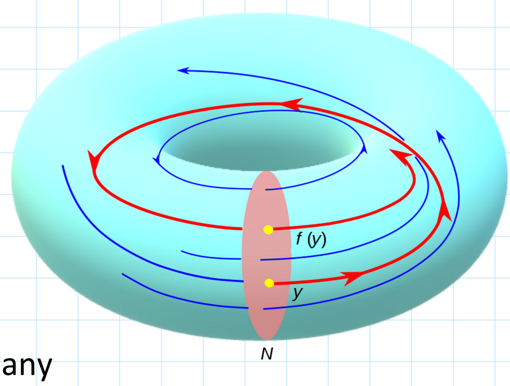
Any confoliation of $S^2 \times S^1$is diffeomorphic to ζ in a C^0-nbh.

$\alpha \wedge d\alpha \geq 0 \; (\leq 0)$.

Proof of the theorem.

Holonomy along a closed curve γ, which is tangent to ξ: the following map $\varphi: I \to I, x \longmapsto y$ ($I \times S^1$ embeds into M, S^1 into γ, I transverse to ξ)

Holonomy φ is:
Nontrivial, if
Linearly nontrivial, if
Attracting/Repelling, if
Sometimes Attracting/Repelling, if



(a) We can C^0 perturb it into a foliation which has only finite many closed leaves.

Def. A minimal set: closed union of leaves which contains no closed union of leaves as a proper subset.

For a foliation after (a), M consists of:
Finite many closed leaves and some exceptional minimal sets (Minimal set which is neither closed leaf nor the entire mfd)
Or, M itself is a minimal set (ξ is minimal).

If M is not minimal: it has linearly nontrivial holonomy (Sachsteder, 1965)
If M is minimal:
(a') Approximate ξ by a fiberation over S^1 (Tischler)
(a'') The fiber is not S^2! Approximate it by foliation with 2 closed leaves.

(b) Thm. (M, ξ) is C^k- foliation, γ tangent to ξ, has linearly nontrivial holonomy. Then $\exists N \subset N' \subset M$, ξ can be C^k-deformed into a confoliation, which is positive contact in N, unchanged in N'^C, diffeomorphic to ξ in N^C.

(c) If confoliation ξ has contact region $H(\xi)$, and any x connect to $H(\xi)$ by some path tangent to ξ. Then ξ can be deformed into a contact structure.

Thm. Reebless (Taut) foliation ξ approximated by a contact structure ξ', then ξ' is tight (WSSF).
The inverse is not always true!

(M,ξ) weakly filled by (X,ω): $M=\partial X$, $\omega(v,w)>0$.
(M,ξ) strongly filled by (X,ω): $N=\partial X$, $\exists$ Dialating vector field v near ∂X:
$\mathcal{L}_v\omega=\omega$, (Then $\alpha\coloneqq\iota_v\omega$, $d\alpha=\omega$ and $\alpha\wedge d\alpha=\frac{1}{2}\iota_v(\omega\wedge\omega)$ is volume form.)
And $\xi=\mathrm{Ker}\,\alpha$.

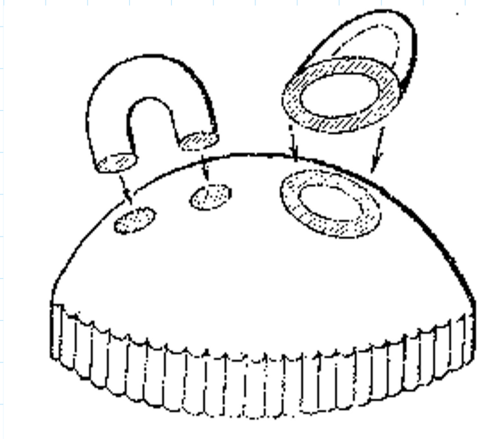
Strongly filled $\Longrightarrow$Weakly filled.

Thm. (X,ω) weakly fills (M,ξ), then $\exists(X,\omega')$ strongly fills it.

Thm. (X_1,ω_1) (convex) strongly fills (M,ξ), (X_1,ω_1) strongly fills (M,ξ) with vector field points into X (Concave strongly). Then
$X=X_1\cup_M X_2$ has symplectic form ω, $\omega|_{X_1}=\omega_1$, and away from a nbh of ∂X_2, $\omega|_{X_2}=c\omega_2$.

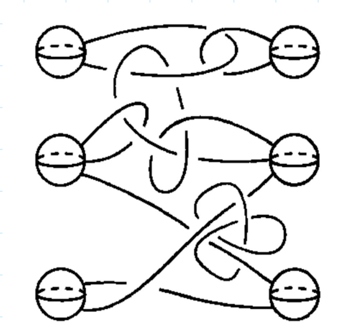
Attaching handles!

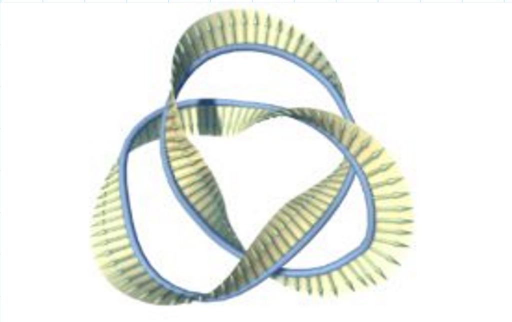
k-handle attached to n-mfd: A copy of $D^k\times D^{n-k}$ attached to ∂X along $\partial D^k\times D^{n-k}$.



Handle decomposition of a (closed, connected) 4-mfd:
- A 0-handle.
- Some 1-handles: Connect a pair of balls (In S^4) to each other.
- Some 2-handles: Attach along some thickened knots in ∂X_1, with framing.
- 3-handles and 4-handles are uniquely determined.

Kirby diagram:





Thm (Elishaberg, Weinstein) (X,ω) with strong or weak convex boundary.
X' is derived by:
- Attaching 1-handles to X, or
- Attaching 2-handles: A knot K, $T_xK\subset\xi_x$. Normal bundle of ξ in TM is a cacnonical

framing (Contact framing). K: Lagrangian knot.
Attaching the 2-handle with framing 1 less than the contact framing.
Then the symplectic form extended to X', and the new boundary is still strong/weak convex!

Thm (Elishaberg, Etnyre) Compact symplectic mfd (X, ω) with weak boundary can embed into a closed symplectic mfd (X', ω').