Brief intro to Legendrian and Transverse knots

Recall: A contact structure ξ on $\mathbb{R}^3$ is a 2-plane field given locally as $\ker\alpha$ where α is a 1-form satisfying $\alpha\wedge d\alpha\neq 0$.

Ex: 1) $\xi_{std}=\ker(dz-y\,dx)$ $=\mathrm{span}\{\partial_z,\ \partial_x+y\partial_z\}$

2) $\xi_{sym}=\ker(dz+r^2d\theta)$ $=\mathrm{span}\{\partial_r,\ r^2\partial_z-\partial_\theta\}$

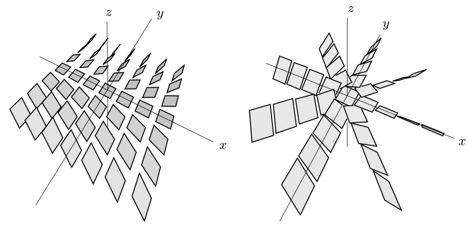


FIGURE 1. The contact structure ξ_{std} (left) and ξ_{sym} (right) on $\mathbb{R}^3$. (Figures courtesy of S. Schönenberger.)

from Etnyre's Leg. and Transversal Knots

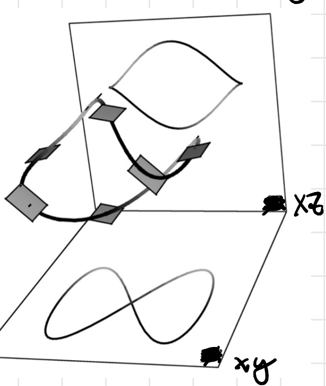
Given a contact structure ξ on $\mathbb{R}^3$, an embedding $i: S^1\longrightarrow\mathbb{R}^3$ we have 3 cases:

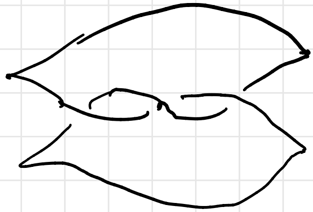
1. $i(S^1)$ is tangent to ξ $\longrightarrow$ $(T_x\, i(S^1)\subset\xi_x\ \ \forall x)$
2. $i(S^1)$ is transverse to ξ $\longrightarrow$ $(T_x\, i(S^1)\oplus\xi_x=T_x\mathbb{R}^3\ \ \forall x)$
3. $i(S^1)$ is sometimes tangent, sometimes transverse

We call 1 a Legendrian knot and 2 a transverse knot

1

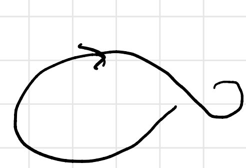
(a)





from Joshua Sabloff's 'What is a Leg. Knot' in AMS notices
https://www.ams.org/notices/200910/tx091001282p.pdf

2



Equivalence of links:

Def: A Legendrian (Transverse) isotopy is an isotopy through a family of Leg. (Transverse) links

Fact: $\exists$ Knots that are smoothly isotopic but not Leg. (transversely) isotopic

Fact: Any smooth link can be C^0 approximated by a Legendrian link

Legendrian links (in $\mathbb{R}^3, \xi_{std}$)

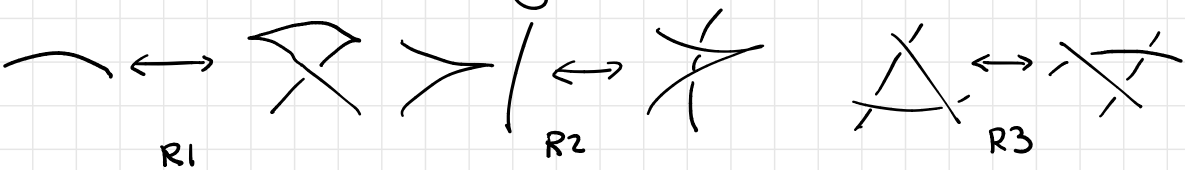
Front projection $\Pi : \mathbb{R}^3 \to xz$-plane

Lagrangian projection $\tilde{\Pi} : \mathbb{R}^3 \to xy$-plane

Properties of Front projections:

- no vertical tangencies $dx = 0 \Rightarrow dz = 0$
- recover y coord. by $y = \frac{-dz}{dx}$
- slope of overcrossing more negative

- Reid. moves given by

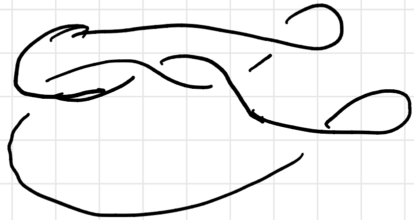


Properties of Lagrangian Projections

- recover z-coord by $z = z_0 + \int_0^{\theta} y(\theta)\, x'(\theta)\, d\theta$
- Must satisfy

(1) $\int_0^{2\pi} y(\theta)\, x'(\theta)\, d\theta = 0$

(2) $\int_{\theta_1}^{\theta_2} y(\theta)\, x'(\theta)\, d\theta \neq 0$ for θ_1, θ_2 corresp. to a crossing



- Partial Reidemeister moves:

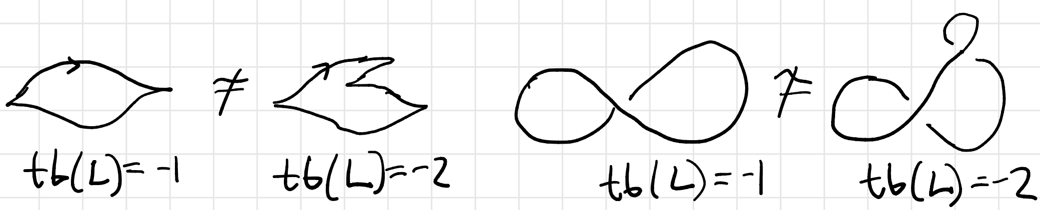
K₁ and K₂ are Legendrian isotopic only if their Lagrangian projections are related by L2 and L3.

Classical invts of Leg. links

- Thurston-Bennequin # - in $(\mathbb{R}^3, \xi_{std})$ given by linking number of L with small pushoff in ∂_z direction.

$tb(L) = \text{writhe}(\pi(L)) - \frac{1}{2}(\#\text{ of cusps}) = \text{writhe}(\pi(L))$

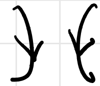
Ex:



Rotation # $r(L) = \frac{1}{2}(D - U) = \text{winding}\, \pi(L)$

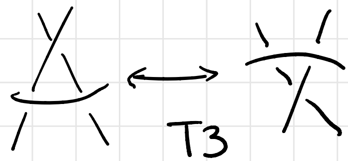
Transverse links

Front projection:

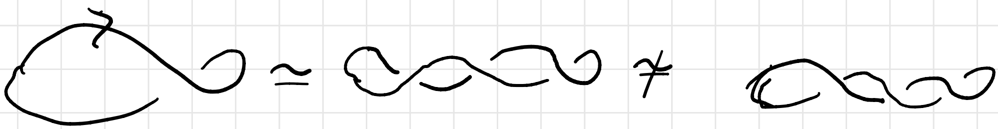
- no downward vertical tangencies 
- no crossings of the form 

Thm (2.9 in Etnyre) Any diagram satisfying the above 2 conditions gives a transverse knot in $(\mathbb{R}^3, \xi_{std})$. Two diagrams represent the same transverse isotopy class if and only if they are related by the moves below

T2



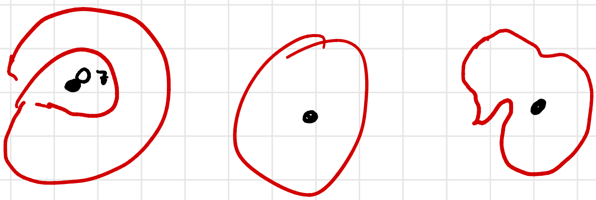
T3

Ex: 

Classical invt:

Self linking $sl(T) = \text{writhe}\, \pi(T)$

Transverse Links in $(\mathbb{R}^3, \xi_{sym})$ $\hookrightarrow$ $\xi_{sym} = \ker(dz + r^2 d\theta$

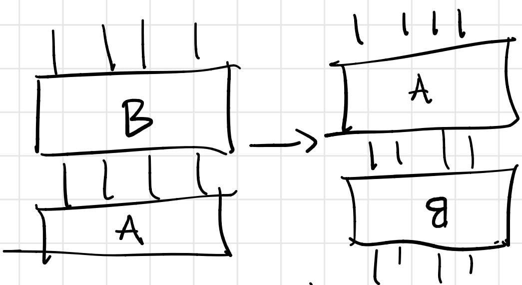
Ex: 

Markov Moves for Transverse Links

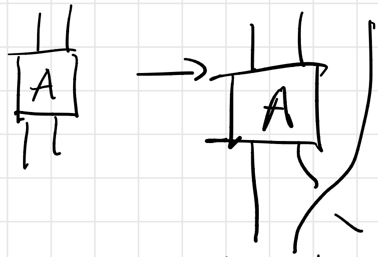
Def: A transverse link $L \in (\mathbb{R}^3, \xi_{sym})$ is a geometric braid if $\partial_\theta|_L > 0$

Thm (Bennequin, '83): Any oriented transvere link is transverse isotopic to the closure of a braid

Thm (Orevkov and Shevchishin '02): Two braids B_1, B_2 represent transversally isotopic links if and only if we can pass from B_1 to B_2 by conjugation, positive Markov moves and inverses



Conjugation (M1)



Pos. Stabilization (M2+)

Notation:

- $S := S^1 \sqcup \dots \sqcup S^1$
- Given $\mathcal{L} : S \times I_t \to (\mathbb{R}^3, \xi_{sym})$, write $L_t = \mathcal{L}(\cdot, t)$

Def: A transversal isotopy $\mathcal{L} : S \times I_t \longrightarrow \mathbb{R}^3$ is monotone near the axis if $\exists\, t_1 < \dots < t_k \in I$ such that:

1) $\forall t_i\ \exists!\, s_i \in S$ such that $\mathcal{L}^{-1}(O_z) = \{(s_1, t_1), \dots, (s_k, t_k)\}$

2) In every nbhd of (s_i, t_i), $\mathcal{L}$ is given by
$x = \tau - 3s^2$, $y = s\tau - s^3$, $z = z_i + s$
for τ coordinate on I centered at t_i and s a coord. on S centered at s_i

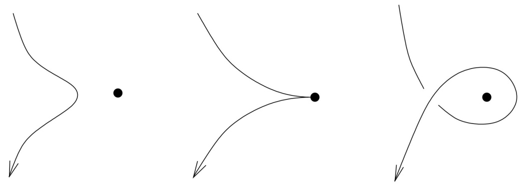


FIGURE 1. THE CURVE $s \mapsto (\tau - 3s^2, s\tau - s^3)$

$\mathcal{L}$ is monotone everywhere if L_t is a geometric braid for $t \notin \{t_1, \dots, t_k\}$ and monotone near the axis

Goal: Make every isotopy monotone everywhere

Note: Fig. 1 represents a positive stabilization

Steps:

1) Show $\mathcal{L}$ can be perturbed so as to be monotone near the axis

2) Upgrade $\mathcal{L}$ to be monotone everywhere

1) Replace every small nbhd of $p=(s_i,t_i)\in\mathcal{L}^{-1}(O_z)$ by fig 1. As long as U is sufficiently small, we can ensure that L_t is transrverse.

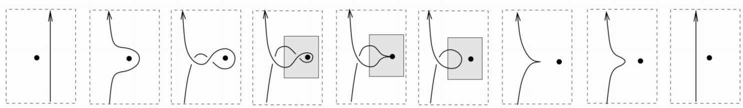


Figure 2. Making the isotopy monotone near Oz

Specifically, $\frac{\partial z}{\partial s}>\varepsilon$ near O_z and can choose U small enough that $r^2\frac{\partial\theta}{\partial s}<\varepsilon$ so that $\frac{\partial z}{\partial s}-r^2\frac{\partial\theta}{\partial s}\Big|_{L_t}>0$

2) Need to make L_t a braid for $t\neq t_i$

Def: A bad zone of $\mathcal{L}$ is anywhere $\mathcal{L}$ is not a braid i.e. any connected component in $S\times I$ s.t. $\partial\theta|_{L_t}\leq 0$

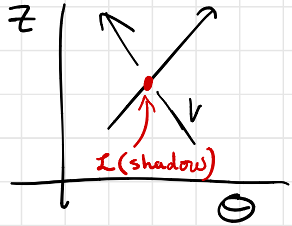
We call a bad zone V simple if

(1) $V_t:=(S\times t)\cap V$ is connected for all $t\in I$

(2) The total increment of θ along V is less than 2π

The shadow of $\mathcal{L}$ on a bad zone V is the set

$$\{(s_0,t_0)\in V \mid \text{the shortest segment between } \mathcal{L}(s_0,t_0) \text{ and } O_z \text{ intersects } \mathcal{L}(\cdot,t_0) \text{ at } (s_0,t_0)\}$$



Equivalently, points where $\mathcal{L}(V)$ is an undercrossing in the θz-projection

Lemma: Can eliminate simple non-shadowed bad zones

Note: z-coord fixed

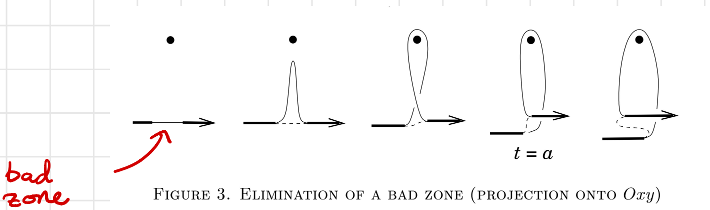


FIGURE 3. ELIMINATION OF A BAD ZONE (PROJECTION ONTO Oxy)

Lemma: Can "wrinkle" a bad zone in a small nbhd U of a smooth curve $l\in V$ so that $\varepsilon > \frac{\partial\theta}{\partial s}\Big/\frac{\partial z}{\partial s} > 0$ in that nbhd

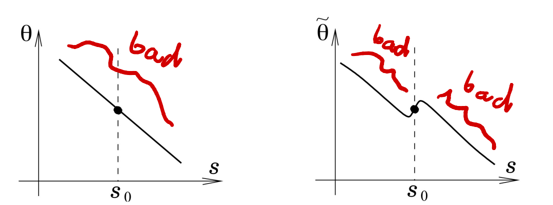


FIGURE 4. WRINKLING

Morally: cut a bad zone along a smooth curve $l\in S\times I$

Need to examine singularities of projection of $\mathcal{L}$ onto θz-cylinder:

0. Generic singularities $\longleftrightarrow$ crossings
1. L_t meets z-axis (as in fig 1)
2. L_t has a unique ordinary tangency pt (T 2)
3. L_t has a unique triple pt (T3)

A sing. is positive if $\frac{\partial \theta}{\partial s} > 0$ for every pt of L_t projecting onto it and non-positive o/w.

A sing. is bad if there is a negative arc shadowed by some other arc

Lemma: We can perturb all bad non-positive singularities of type (2) and (3)

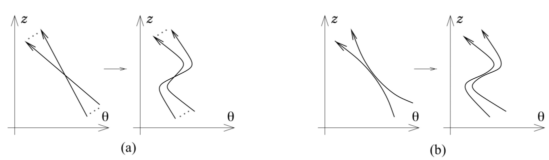
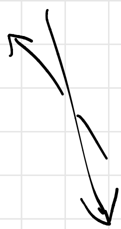


FIGURE 5. ELIMINATION OF BAD NON-POSITIVE SINGULARITIES

bad zones shadowed by bad zones

impossible



Algorithm:

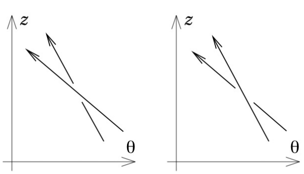
Given L monotone near the axis with bad zones $V_1, \dots, V_n$ we eliminate bad zones successively via the following steps.

1) Eliminate bad non-positive singularities of Type (2) and (3)

Denote the shadows of V_i on V_1 by $l_1, l_2, \dots l_k$

2) Wrinkle along components of bad zones V_i shadowing V_1 as in fig. 6b

3) Wrinkle V_1 wherever it's shadowed to get non shadowed bad zones → corresponding to V_1



(a) (b)

FIGURE 6.

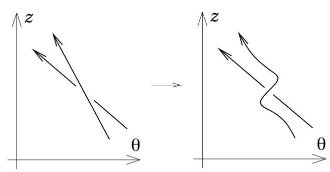


FIGURE 7. WRINKLING AT STEP 2

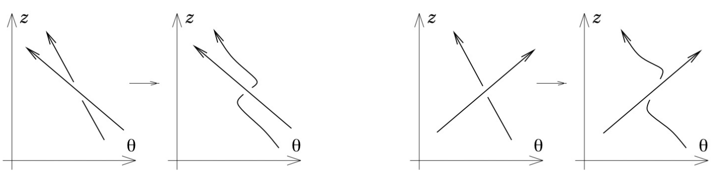


FIGURE 8. WRINKLING AT STEP 3

4) Wrinkle new bad zones if needed to make sure they're simple

5) Apply fig. 3 to get rid of non-shadowed bad zones.

6) Repeat for successive V_j

Note: At each step, we wrinkle away from tangencies and triple pts, so we can make sure no new shadow appears.