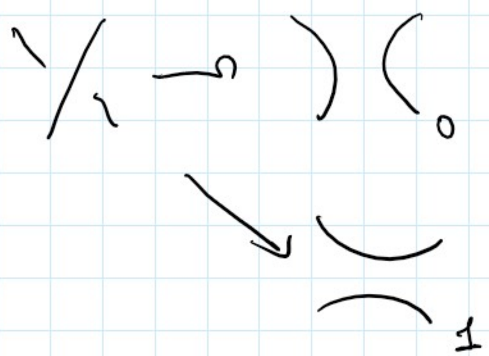
Recap on Khovanov homology

Wednesday, October 7, 2020 11:23 AM

D. Bar-Natan. On Khovanov's categorification of the Jones polynomial.

① Cube of resolutions



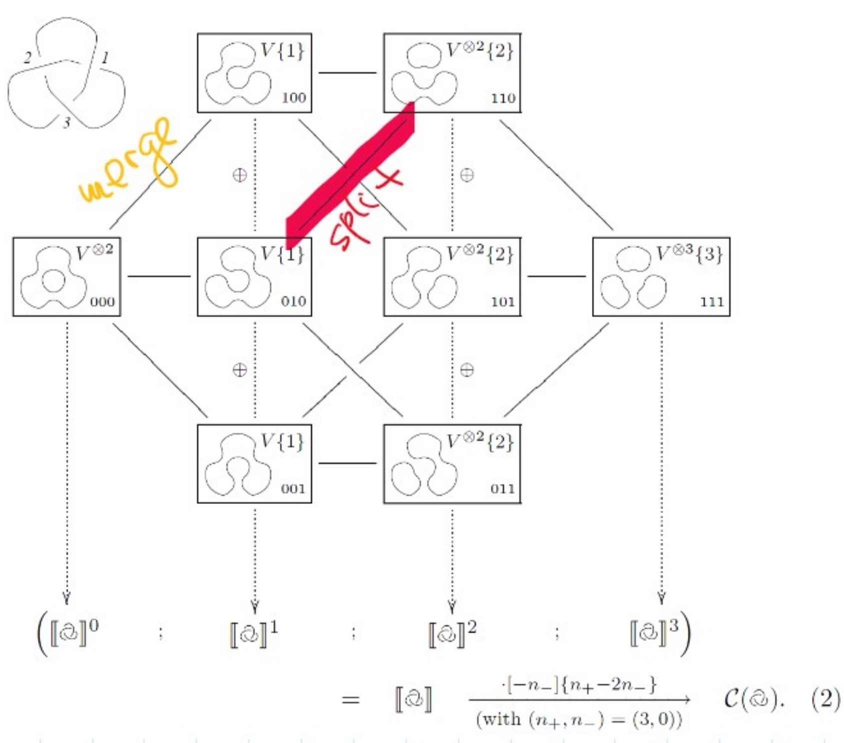


Diagram with r crossings $\Rightarrow$ 2^r resolutions
= vertices of r-dimensional cube $\{0,1\}^r$

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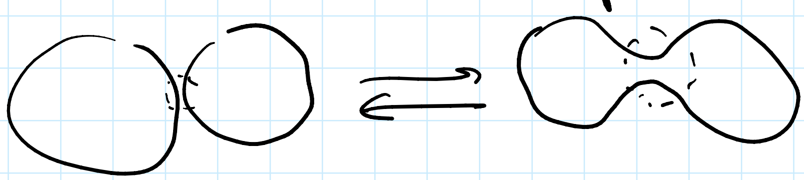
Vertices of the cube of resolutions
$\Rightarrow$ collections of disjoint circles.

Edges: swap one 0 with 1,
keep all other 0's and 1's unchanged.

That is, change the resolution of one crossing from 0 to 1, keep the rest unchanged.



Either merge two circles into 1
or split a circle into 2.

② Algebra $A = \mathbb{Q}[x]/(x^2)$

A is a two dimensional algebra with basis $1, x$ and multiplication

$$m : A \otimes_{\mathbb{Q}} A \longrightarrow A$$

basis in $A \otimes A$:

$$\begin{cases} 1 \otimes 1 \longrightarrow 1 \\ x \otimes 1 \longrightarrow x \\ 1 \otimes x \longrightarrow x \\ x \otimes x \longrightarrow 0 \ (= x^2) \end{cases}$$

$m: A\otimes A \to A \quad \begin{cases} 1\otimes x \longrightarrow x \\ x\otimes x \longrightarrow 0 \; (= x^2) \end{cases}$

Comultiplication:

$\Delta : A \longrightarrow A \otimes A$

$1 \longrightarrow 1\otimes x + x \otimes 1$

$x \longrightarrow x \otimes x$

__Rmk__ A has a nondegenerate bilinear form, we can use it to identify $A \simeq A^*$

$m: A\otimes A \longrightarrow A \qquad m^*: A^* \longrightarrow A^* \otimes A^*$ same as Δ.

$(1, x) = (x, 1) = 1 \qquad (1,1) = (x, x) = 0$

$(a,b) = $ coef. at x of $a \cdot b$.

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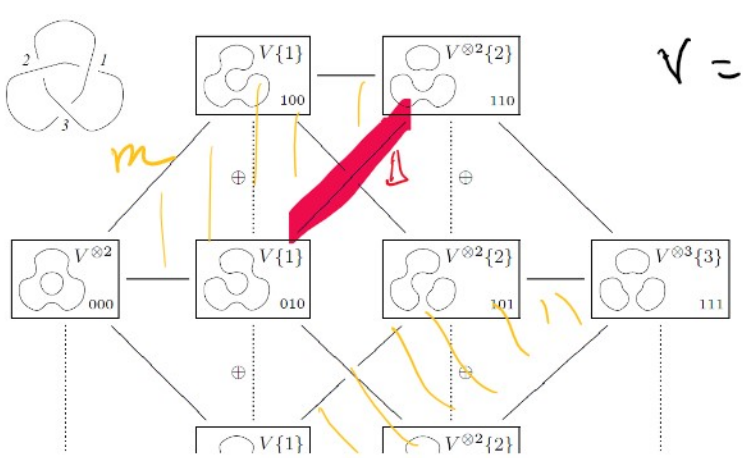
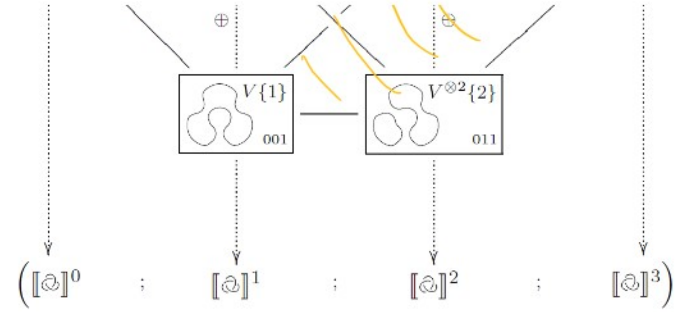
(3) Key idea: associate $A^{\otimes m}$ to a resolution of our link with m circles.

Vertices $\Rightarrow A^{\otimes m}$



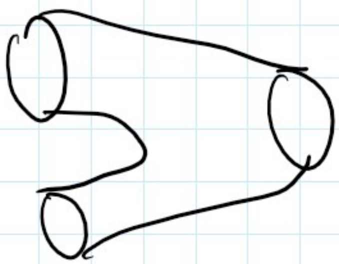
$V =$ our A



$$\Big([\![\,\cdot\,]\!]^0 \quad ; \quad [\![\,\cdot\,]\!]^1 \quad ; \quad [\![\,\cdot\,]\!]^2 \quad ; \quad [\![\,\cdot\,]\!]^3 \Big)$$

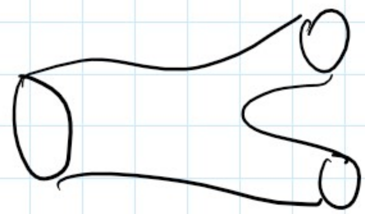
$$= \quad [\![\,\cdot\,]\!] \quad \xrightarrow[\text{(with } (n_+, n_-) = (3,0))]{\cdot[-n_-]\{n_+ - 2n_-\}} \quad \mathcal{C}(\cdot). \quad (2)$$

Edges ⇒ merge / split of 2 circles



2 circles ⇒ 1 circle
0 resolution 1-resolution.

$m : A \otimes A \longrightarrow A$



1 circle 2 circles
0-resolution 1-resolution.

$\Delta : A \longrightarrow A \otimes A$

Edges ⇒ linear maps between spaces at vertices.

$A \otimes A \otimes A^m \longrightarrow A \otimes A^m$

other circle

$m \otimes \mathrm{Id}$.

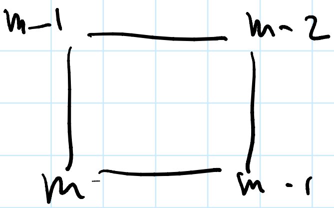
<u>Lemma</u>: All 2-dimensional faces commute.

<u>Proof</u>: Several checks

$m-1$ —— $m-2$

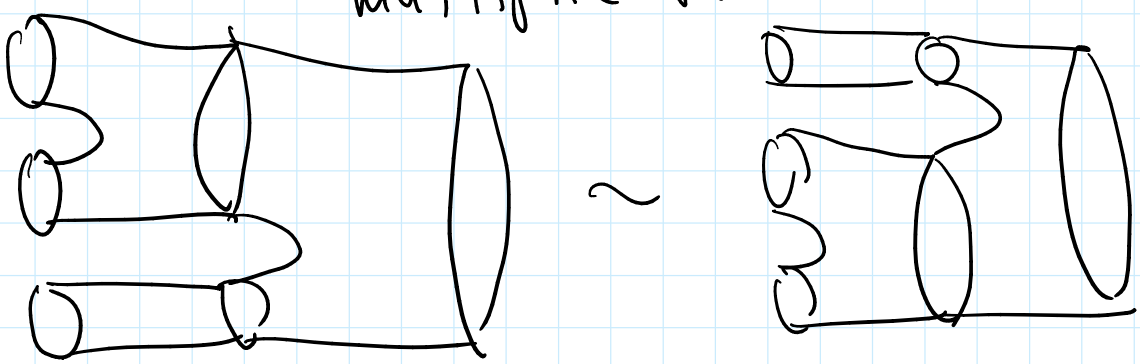
$A \otimes A \xrightarrow{m} A$

$ab \otimes c \longrightarrow a\cdot(bc) = (ab)\cdot c$

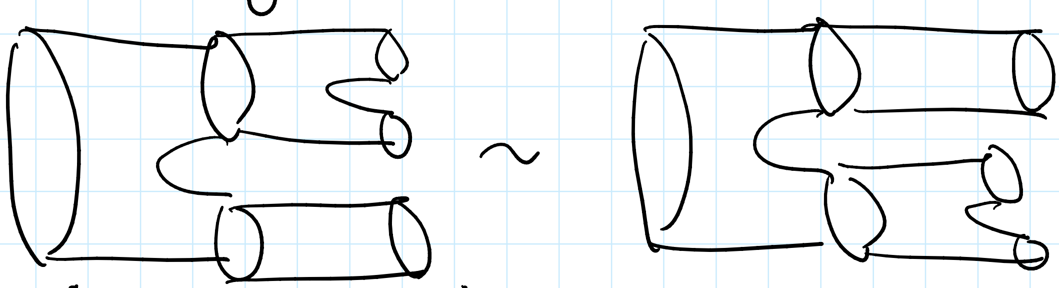


$$\begin{array}{ccc} A\otimes A & \longrightarrow & A \\ \uparrow m_{12} & & \uparrow m \\ A\otimes A\otimes A & \xrightarrow{m_{23}} & A\otimes A \end{array} \qquad \begin{array}{ccc} ab\otimes c & \longrightarrow & a\cdot(bc) \\ \uparrow & & \uparrow \\ a\otimes b\otimes c & \longrightarrow & a\otimes bc \end{array}$$

This is associativity of multiplication!

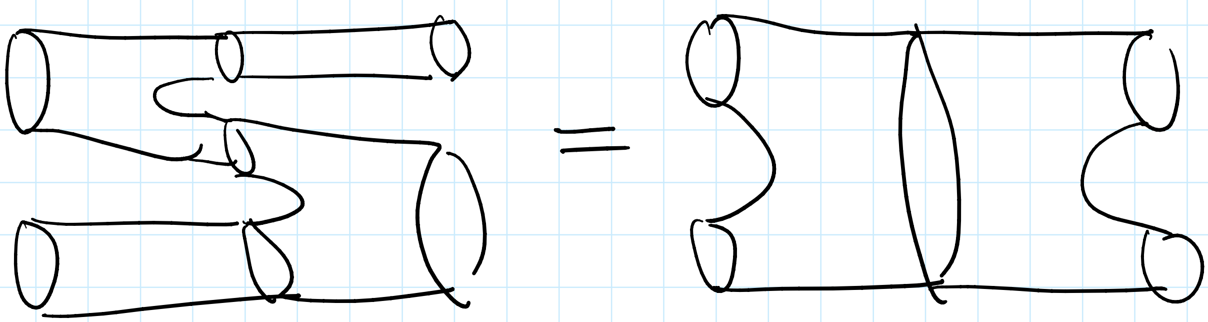


Coassociativity



$$\begin{array}{ccc} A\otimes A & \xrightarrow{\Delta_1} & A\otimes A\otimes A \\ \Delta \uparrow & & \uparrow \Delta_2 \\ A & \xrightarrow{\Delta} & A\otimes A \end{array}$$

Frobenius relation



$A\otimes A\otimes A \xrightarrow{m_{23}} A\otimes A$

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m_{23}} & A\otimes A \\ \Delta_1 \uparrow & & \uparrow \Delta \\ A\otimes A & \xrightarrow{m} & A \end{array}$$

commutative diagram.

(this is a check).

Works for our choice of m, Δ. (Exercise).

Rmk We can look at "1+1 dim TQFT"

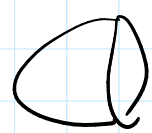
Objects = collections of circles

Maps = cobordisms between them / isotopy.

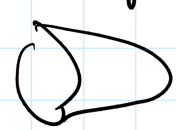
The above relations generate all isotopy.

Need two more maps

+ some relations for those.



$\emptyset \longrightarrow 1$ circle, i



circle $\longrightarrow \emptyset$, ε

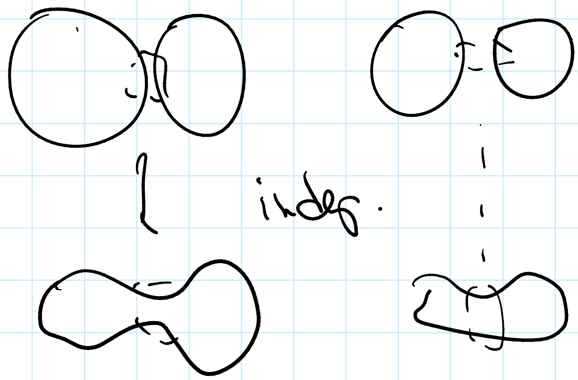
An algebra A with $m: A\otimes A \to A$, $\Delta: A \to A\otimes A$ is called a Frobenius algebra, if it is commutative, cocommutative, assoc., coassoc.

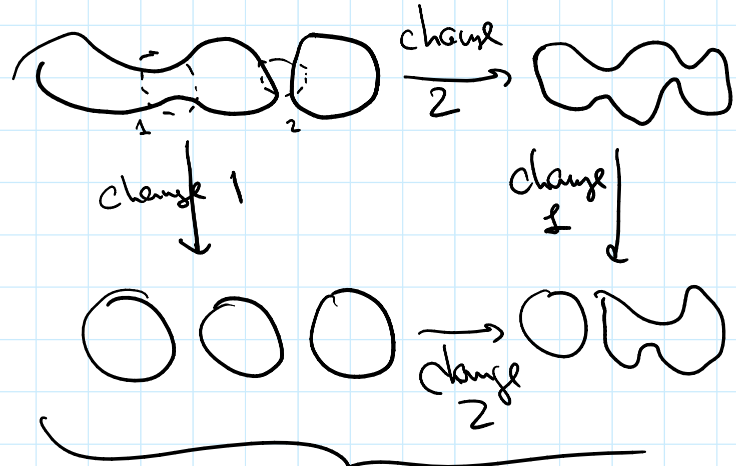
Frobenius relation, $i: \mathbb{Q} \longrightarrow A$ unit $1 \to 1$

$\varepsilon: A \to \mathbb{Q}$ counit $1 \to 0$, $x \to 1$

Rmk There are lots of Frobenius algebras.

2d faces of the cube = pairs of crossings.





Frobenius relations.

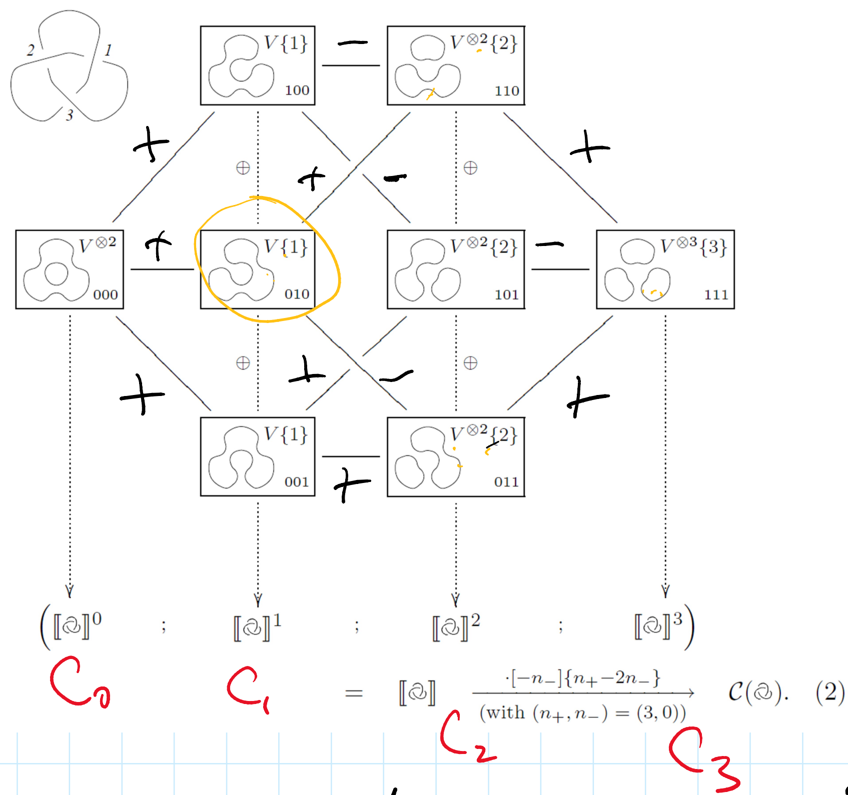
Go through all possible cases for pairs of resolutions $\Longrightarrow$ easy commutation, or assoc/coass. or Frobenius relation.

④ Khovanov complex:

$$\to C_i \xrightarrow{d} C_{i+1} \xrightarrow{d} C_{i+2} \to \cdots$$

$C_i = \bigoplus$ (all vector spaces at vertices v with $|v| = i$)

that is, v has i ones
$n-i$ zeroes.



d = sum of edge maps with signs

edge changes some 0 to 1
sign $= (-1)^{\#1 \text{ before this } 0}$

In each 2d square we have odd number of minuses.

$\Longrightarrow d^2 = 0$

<u>Def</u> Khovanov homology of L = homology of this chain cpx

<u>Thm</u> (Khovanov) This is actually

Thm (Khovanov) This is actually a topological link invariant, does not depend on a diagram.

Proof Prove it does not change under Reidemeister moves.

~~Homology is actually bigraded:~~

- Homological grading $= |v|$ number of ones (up to overall shift).
- q-grading (or quantum): use grading on A

$\deg_q(1) = 0 \quad \deg_q(x) = 2$

This defines grading on $A^{\otimes n}$

$\deg_q(a \otimes b \otimes c) = \deg_q(a) + \deg_q(b) + \deg_q(c)$

$m : A \otimes A \to A$ preserves the grading.

$\Delta : A \to A \otimes A$ changes this grading by 2

$0 \quad 1 \to 1 \otimes x + x \otimes 1 \quad 2$

$2 \quad x \to x \otimes x \quad 4$

$2 \quad x \longrightarrow x \otimes x \quad 4$

One can adjust this grading by $|v|$ and #circles such that the differential d preserves q-grading.

$\Rightarrow$ all homologies are bigraded by homological & q-grading.

$H_{i,j}$ = homology in hom. degree i quantum degree j

C_{ij} = chain groups — " —

<u>Observe</u>: $\sum (-1)^i \dim H_{ij}\, q^j =$

$= \sum (-1)^i \dim C_{ij}\, q^j$

can <u>compute</u>!

$= \sum_{\text{vertices}} (-1)^{|v|} \left(\text{graded dimension of } A^{\otimes m}\right) \cdot q^{\cdots}$

$= \sum_{\text{vertices}} (-1)^{|v|} (1+q^2)^m \cdot q^{\cdots}$

= Jones polynomial of L.

— Jones poly ...

Jones / Kauffman: take same cube of resolutions, sum $(-1)^{|v|} \cdot (q+q^{-1})^m$ over vertices.

Result = Jones poly up to a factor.

⟹ Thm (Khovanov) $\sum (-1)^i \dim H_{ij}\, q^j$ [graded Euler characteristic]

Jones polynomial