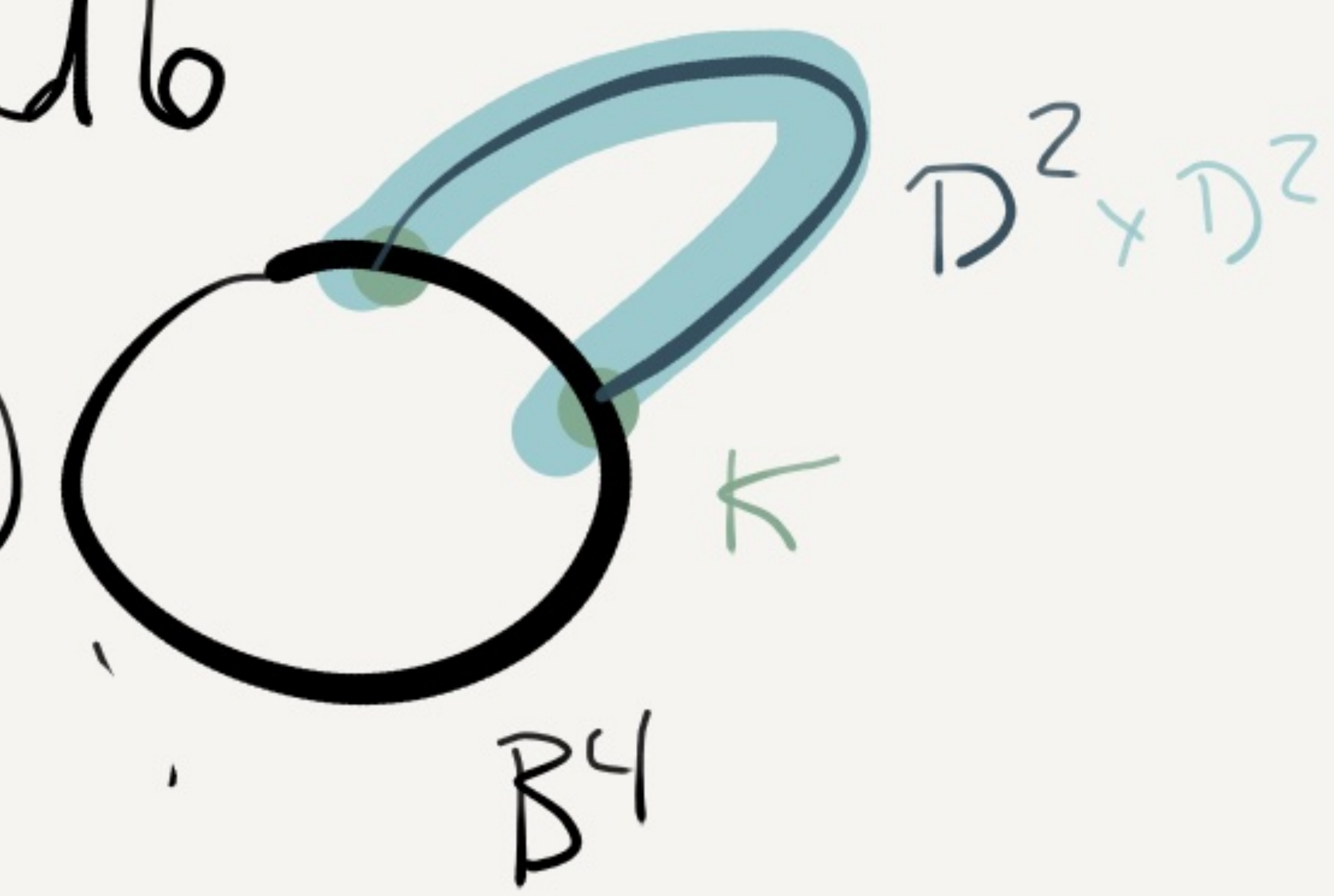
Def^n A **knot trace** $X(K) := B^4 \cup_{(K,0)} 2\text{-handle}$

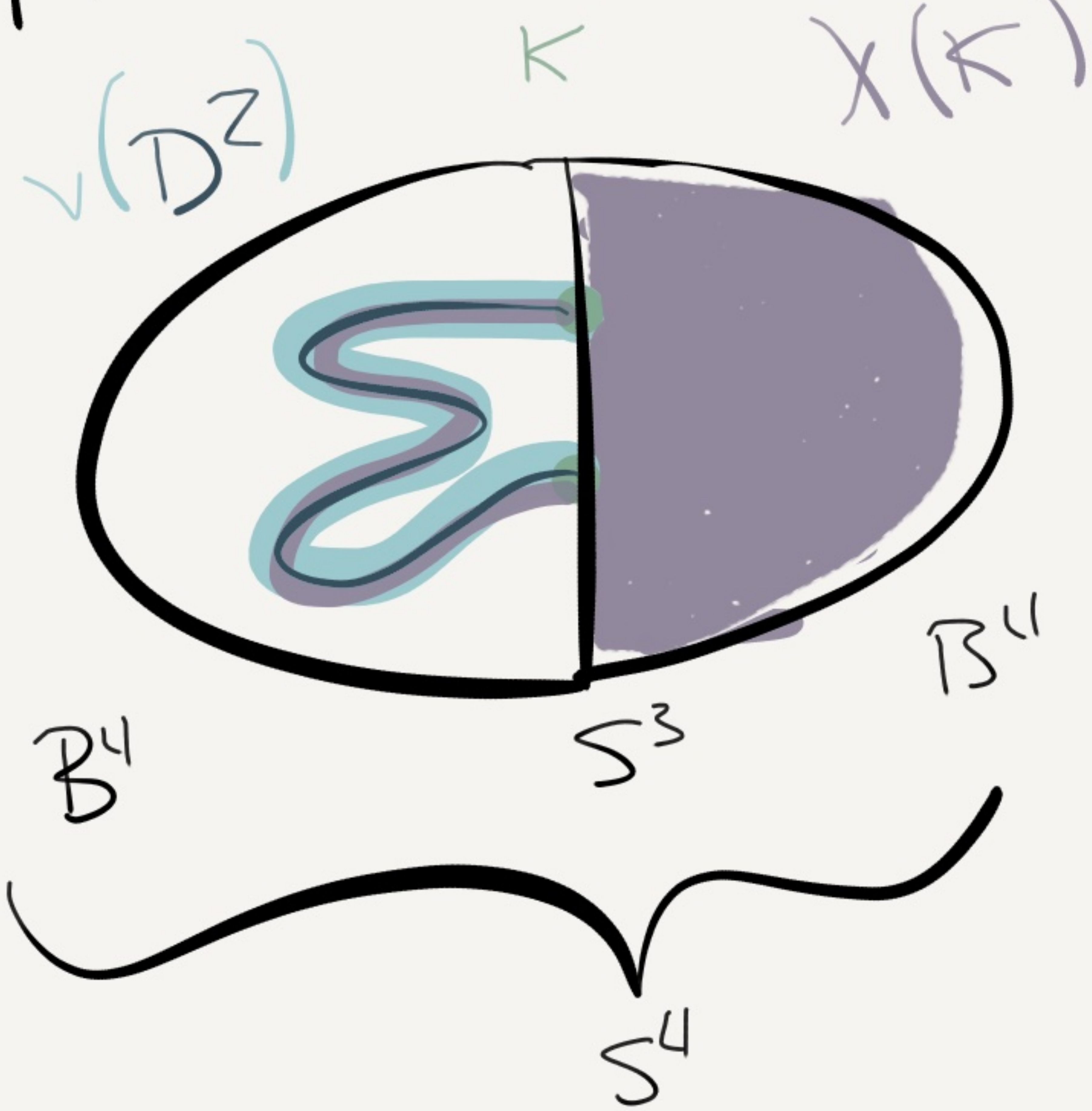


Trace Embedding Lemma: (Fox-Milnor '57)

K is $\overset{\text{top}}{\text{sm}}$ slice $\iff X(K) \underset{\text{sm}}{\overset{\text{top}}{\hookrightarrow}} S^4$

$\iff X(K) \underset{\text{sm}}{\overset{\text{top}}{\hookrightarrow}} B^4$

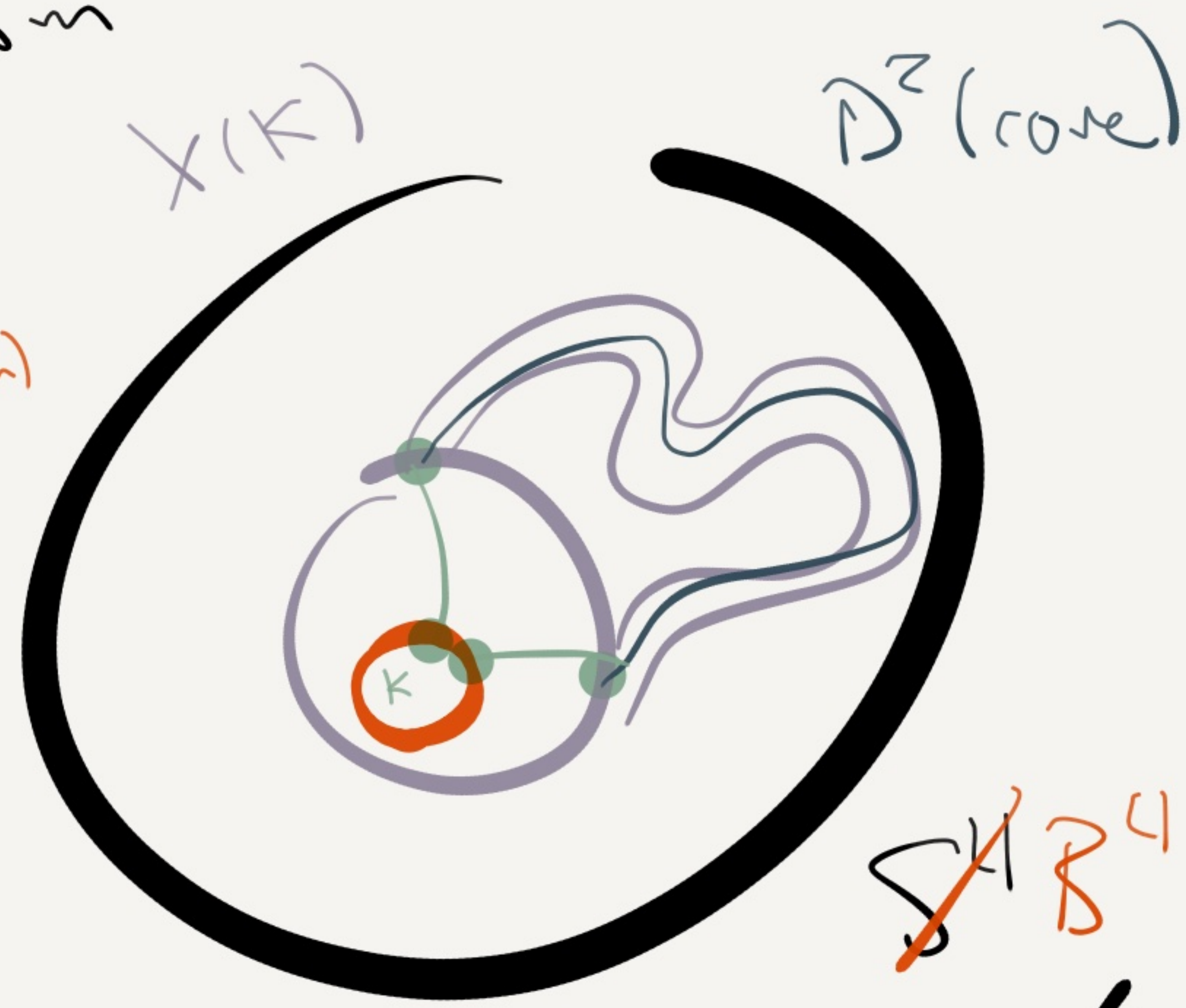
Pf $(\Longrightarrow)$



$(\Longleftarrow)$

~~D^2~~ (PL)
annulus (sm)

~~$B^4 = \nu(\text{cone pt})$~~



//

Thm 1 (Berkeley '80s): $\mathbb{R}^4$ is exotic

Thm 2 (P, '18): The Conway knot is not slice

Sketch 1: Consider K top but not sm. slice.

TEL: $X(K) \xrightarrow[\varphi]{top} \mathbb{R}^4$. $Z := \mathbb{R}^4 \setminus \varphi(\mathring{X}(K))$ ← top mfld noncpct w/ 2

Freedman-Quinn: Z admits sm str Z_{sm}

$W_{sm} := Z_{sm} \cup X(K)$. $W \underset{top}{\cong} \mathbb{R}^4$. $X(K) \underset{sm}{\hookrightarrow} W$.

TEL: $W \underset{sm}{\not\cong} \mathbb{R}^4$ //

Q: Is S^4 exotic?

Attempt: Consider K top slice, not sm slice.

TEL: $X(K) \xrightarrow[\text{top}]{} S^4$ $Z := S^4 \setminus \varphi(\mathring{X}(K))$ ⟵ top mfld cpct w/ ∂

Facts: • many top cpct 4-mfld's not smoothable

• If K is not sm slice via

- classical abelian/metabelian invts
- HF, HFK concordance invts
- Donaldson, 10/8-type conc invts
- adjunction-type sliceness invts

then for every $X(K) \xhookrightarrow[\text{top}]{\varphi} S^4$, Z is <u>not</u> smoothable.

Open Qs • Does $s(K) \neq 0 \Longrightarrow$ any such Z is not smoothable?

• Is some Z for the Conway knot smoothable?

• Find some $K \subseteq \partial W$ s.t. K is slice in W, $s(K) \neq 0$

candidate exotic "B"

<u>Thm Z</u> (P, '18): The Conway knot is not slice

<u>Sketch Z</u>: Build J s.t. $X(J) \cong_{sm} X(\text{Conway})$

TEL $\Longrightarrow$ J slice $\Longleftrightarrow$ Conway slice.

show $s(J) \neq 0$. //

Fact: $\Delta_t(\text{Conway}) = 1 \Rightarrow$ Conway is top slice

Sketch 2: Build J s.t. $X(J) \cong_{sm} X(\text{Conway})$ ①

TEL $\Rightarrow$ J slice $\Leftrightarrow$ Conway slice.

show $s(J) \neq 0$. // ②

③ Is this ever going to work again?

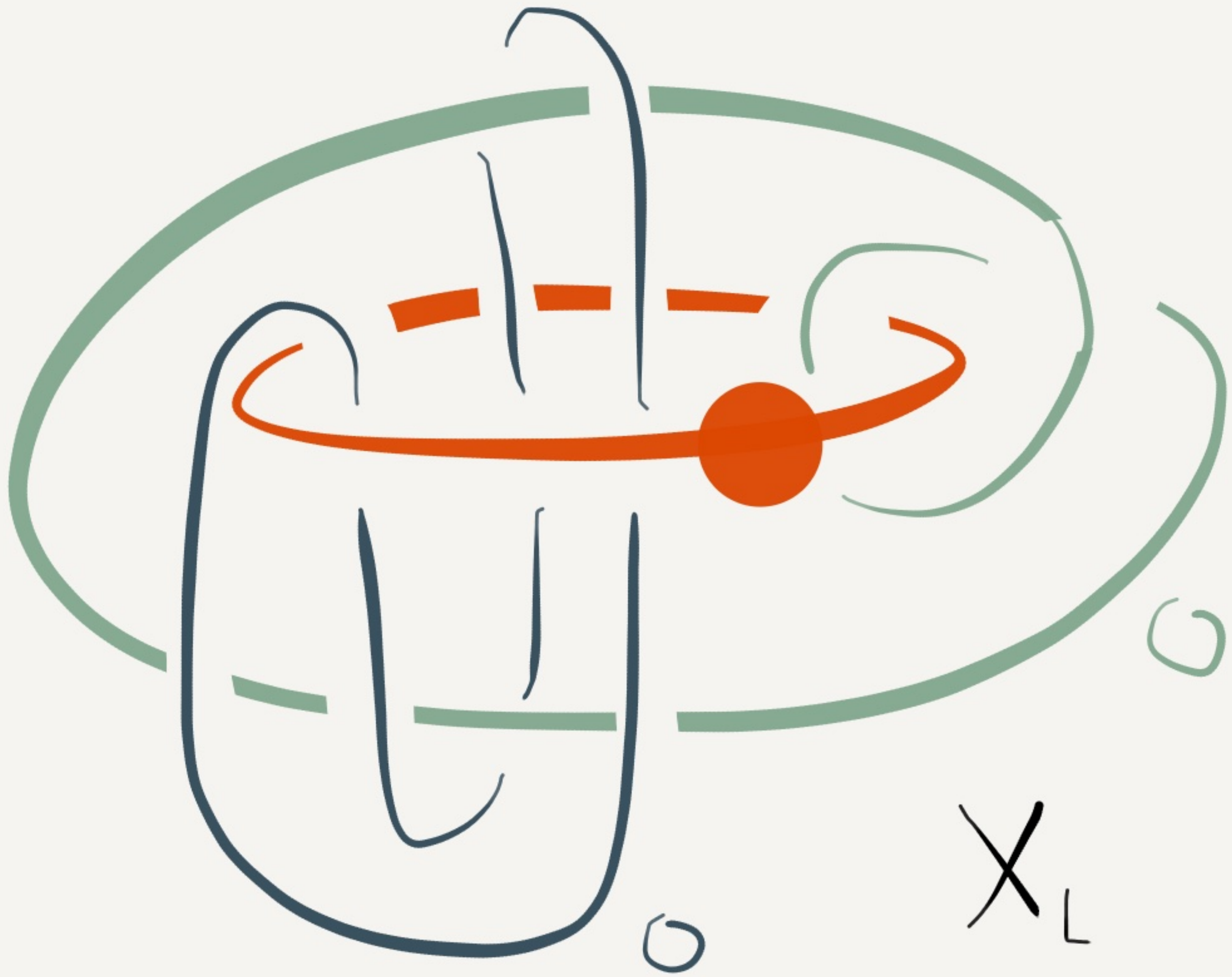
① Build J s.t $X(J) \cong_{sm} X(\text{Conway})$

RBG construction

$L = R \cup B \cup G$ s.t

- $R \cup B \cong M_B \cup B$
- $R \cup G \cong M_G \cup G$
- $lk(B, G) = 0$

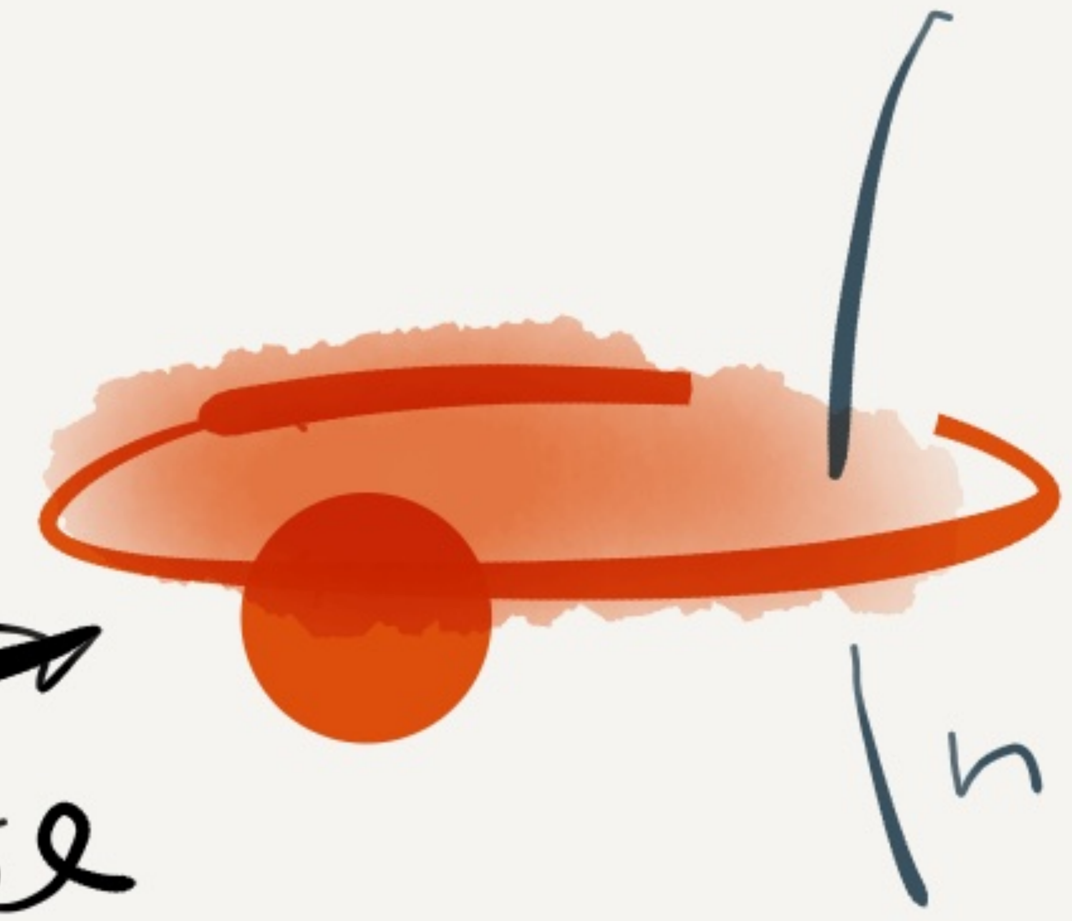
$L \rightsquigarrow X^4_L$



X_L

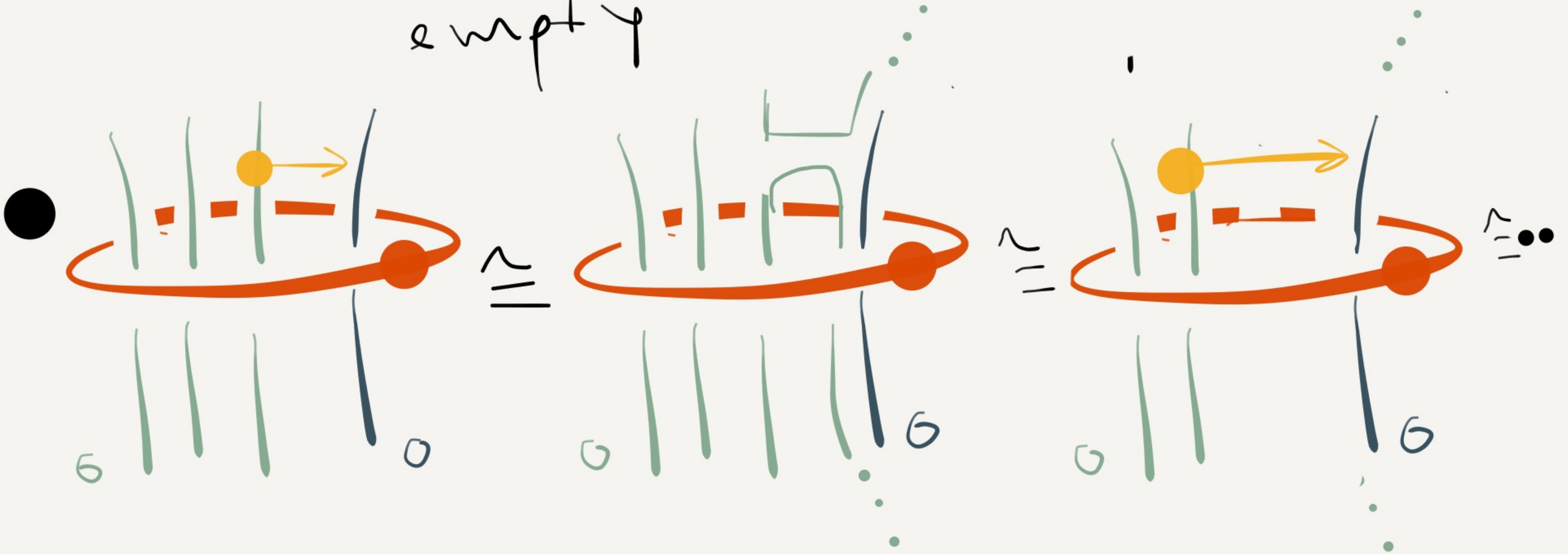
Thm (P): for any such L $\exists$ K_B, K_G s.t $X(K_B) \cong X_L \cong X(K_G)$

Recall: ●



$\cong B^4$

D^2 otherwise empty

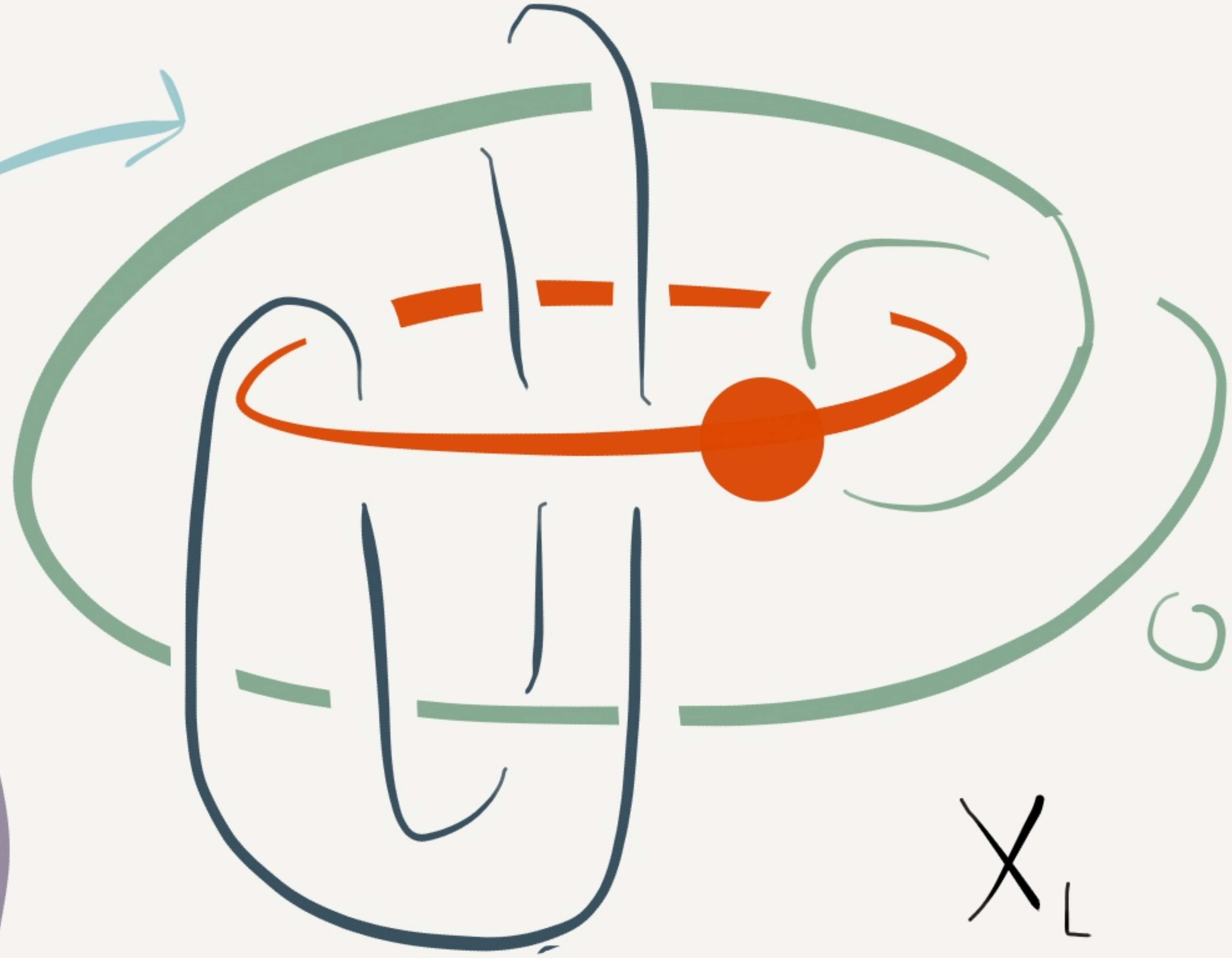


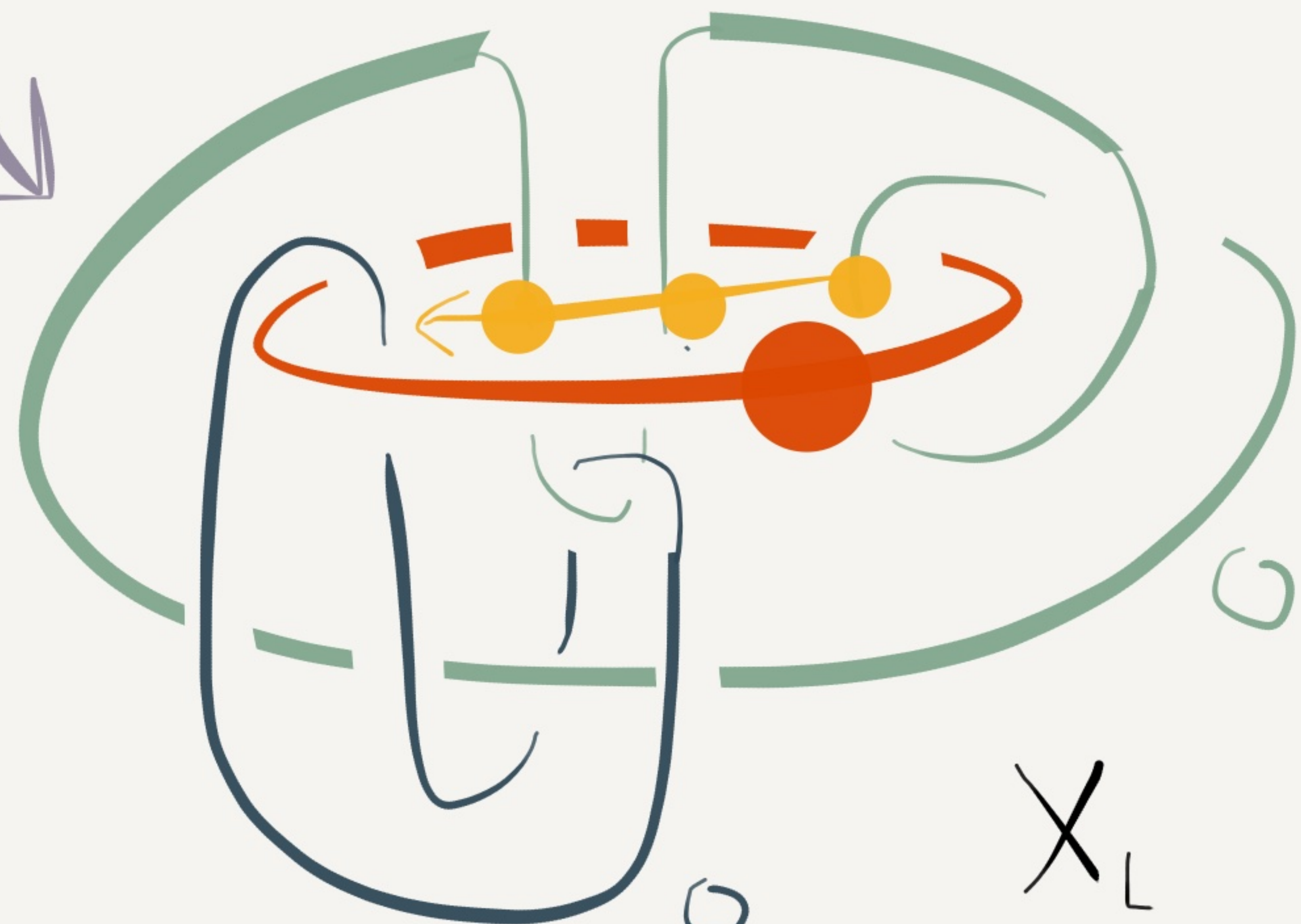
X_L
$\cong$
$\cong$
$=: K_B$

$L = R \cup B \cup G$ s.t

- $R \cup G \cong \mathcal{M}_G \cup G$
- $R \cup B \cong \mathcal{M}_B \cup B$
- $lk(B, G) = 0$

use to get K_G





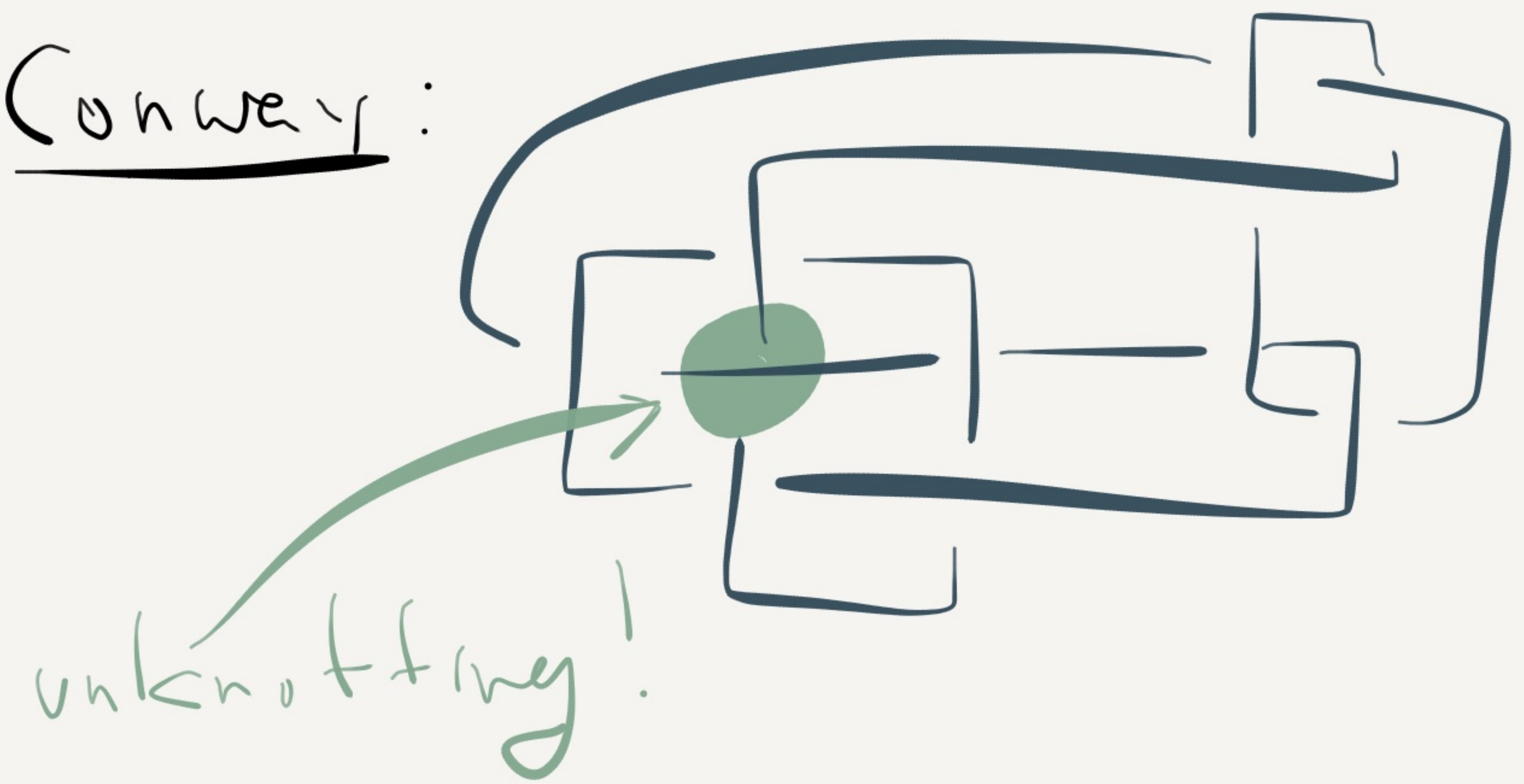
1. Build J s.t. $X(J) \cong_{sm} X(\text{Conway})$

Lemma: If $u(K) = 1$ then $\exists\, L$ w/ $K_B = K$

Fact: not (m)any more such lemmas

Open: $\exists\, J \not\cong T_{2,5}$ w/ $X(J) \cong X(T_{2,5})$

Conway:

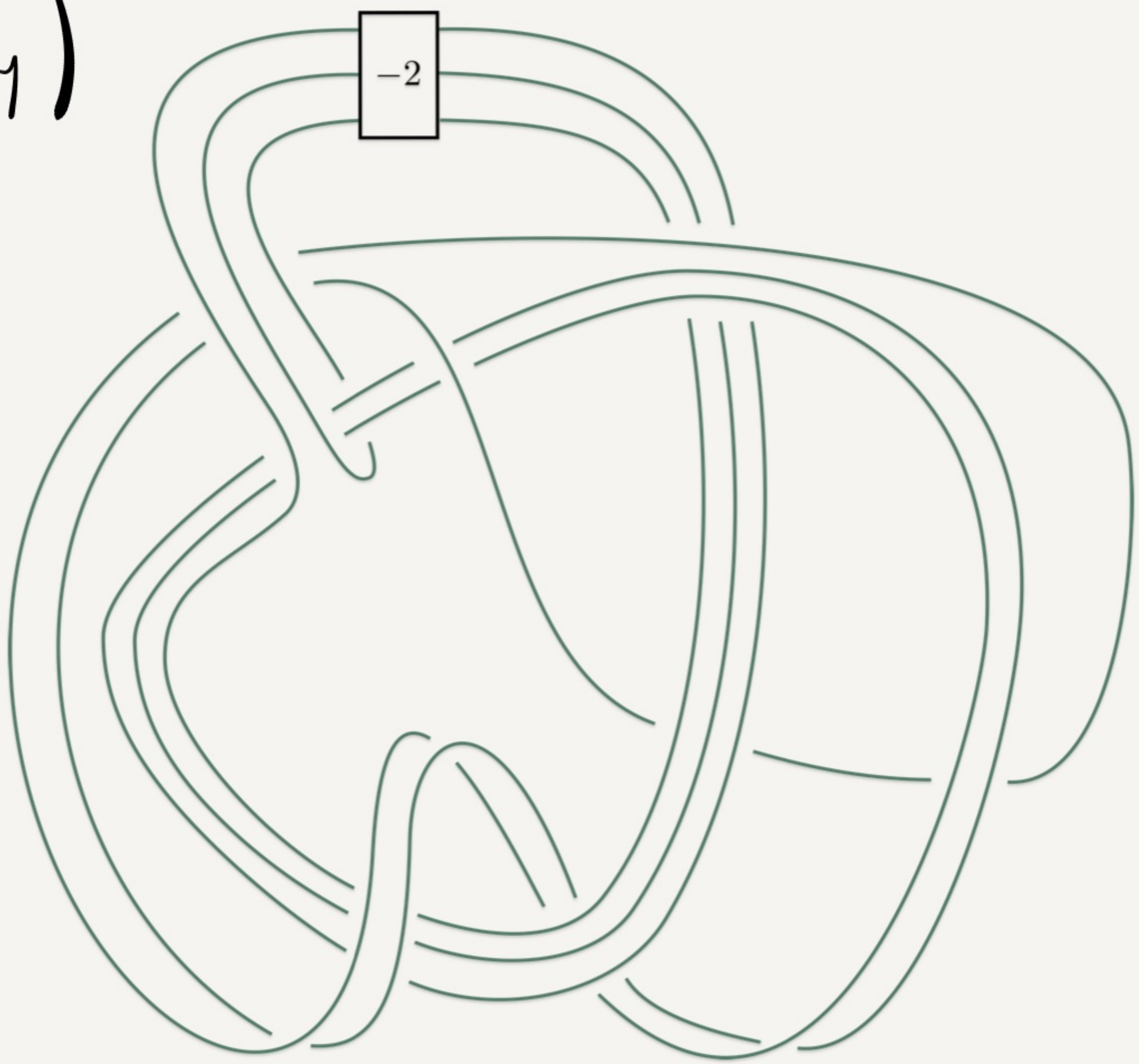


✓ Build J s.t. $X(J) \cong_{sm} X(\text{Conway})$

RBG + Lemma + unknotting crossing gives

$X(K_G) \cong X(\text{Conway})$

$K_G = J =$



show $s(J) \neq 0$. //

(2)

① Draw J in snappy, get DT code

② Mathematica

3. Knot Atlas

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<< KnotTheory`
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.

FindSmallGirthOrdering[
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$$3q^{13} t^6 + q^9 t^7 + 4q^{11} t^7 + 4q^{13} t^7 + 2q^{15} t^7 + 2q^{11} t^8 + 5q^{13} t^8 + 4q^{15} t^8 + [\text{illegible}]\,t^8 + 4q^{13} t^9 + 5q^{15} t^9 + 4q^{17} t^9 + q^{13} t^{10} + 3q^{15} t^{10} + 6q^{17} t^{10} + 3q^{19} t^{10} + 2q^{15} t^{11} + 4q^{17} t^{11} +$$
$$4q^{19} t^{11} + 2q^{21} t^{11} + q^{15} t^{12} + 2q^{17} t^{12} + 6q^{19} t^{12} + 3q^{21} t^{12} + q^{23} t^{12} + q^{17} t^{13} + 4q^{19} t^{13} + 4q^{21} t^{13} + 4q^{23} t^{13} + q^{19} t^{14} + 4q^{21} t^{14} + 4q^{23} t^{14} + 2q^{25} t^{14} + q^{19} t^{15} + 3q^{21} t^{15} + 3q^{23} t^{15} +$$
$$3q^{25} t^{15} + q^{27} t^{15} + q^{21} t^{16} + 4q^{23} t^{16} + 5q^{25} t^{16} + 2q^{27} t^{16} + 2q^{23} t^{17} + 3q^{25} t^{17} + 4q^{27} t^{17} + 2q^{29} t^{17} + 2q^{25} t^{18} + 4q^{27} t^{18} + 2q^{29} t^{18} + q^{31} t^{18} + q^{25} t^{19} + 2q^{27} t^{19} + 4q^{29} t^{19} +$$
$$2q^{31} t^{19} + q^{27} t^{20} + 4q^{29} t^{20} + 3q^{31} t^{20} + 2q^{33} t^{20} + q^{29} t^{21} + 2q^{31} t^{21} + 3q^{33} t^{21} + q^{35} t^{21} + 2q^{31} t^{22} + 2q^{33} t^{22} + q^{35} t^{22} + q^{31} t^{23} + q^{33} t^{23} + 3q^{35} t^{23} + q^{37} t^{23} + q^{33} t^{24} + 2q^{35} t^{24} +$$
$$q^{37} t^{24} + q^{39} t^{24} + q^{35} t^{25} + 2q^{37} t^{25} + q^{39} t^{25} + q^{35} t^{26} + q^{37} t^{26} + q^{39} t^{26} + q^{41} t^{26} + 2q^{39} t^{27} + q^{41} t^{27} + q^{39} t^{28} + q^{41} t^{28} + q^{43} t^{28} + q^{41} t^{29} + q^{43} t^{29} + q^{45} t^{29} + q^{45} t^{30} + q^{45} t^{31} + q^{49} t^{32}$$

gens that survive to KhL live here. $s(J) \in \{0, 2\}$

which?

100%

9:38 AM

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$\longrightarrow t$

Out[6]//MatrixForm=

Recall: spectral sequence differentials are of slope $(1, 4n)$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
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survives!

$s_{max} = 3$

$s = 2$

Conway not slice!

Assuming a matrix | Use as *a list of lists* instead

transpose | matrix plot | display as | matrix rank | more...

$2a$

$f = 0$

E_1

③ Is this ever going to work again?

Open: is $D^+(LHT)$ or $C_{2,1}(f_8 \frac{0}{0})$ slice? ← not ribbon

① Build J s.t. $X(J) \cong_{sm} X(\text{Conway})$

- RBG + $u(K)=1$ lemma
- if $K = D_n^+(J)$ for any n, J then lemma unhelpful ($K_B = K_G$)
- don't have other "if K ___ then $\exists$ RBG w/ $K = K_B$" lemmas

② show $s(J) \neq 0$:

- Kh programs very unreliable over 50 crossings
- we got lucky that could computes from Kh
- most other sliceness invts are $X(K)$ invts

Sketch 2, improved:

① Build J w/ $\partial(X(J))$ $\mathbb{Z}H_*$ cob to $\partial(X(\text{Conway}))$

$$\left(\begin{array}{l} i_*: H_*(\partial X(J)) \to H_*(W) \\ j_*: H_*(\partial X(C)) \to H_*(W) \\ \text{both isoms} \end{array} \right)$$

easier!

A version of TEL $\Longrightarrow$

Conway is slice in some $\mathbb{Z}H_*B^4$ $\Longleftrightarrow$ J is slice in some $\mathbb{Z}H_*B^4$ ✓

② Show J not slice in <u>any</u> $\mathbb{Z}H_*B^4$

harder

② Show J not slice in any $\mathbb{Z}H_*B^4$

~~harder~~ easier

Recall: most sliceness invts are $X(K)$ invts

Fact: most sliceness invts are not $\partial(X(K))$ invts

Recall: If K is not sm slice via ϕ then for every $X(K) \xrightarrow[\text{top}]{\phi} S^4$, Z is not smoothable

Fact: If K is not slice via ϕ then K is not slice in any $\mathbb{Z}H_* B^4$

- classical abelian/metabelian invts
- HF, HFK concordance invts
- Donaldson, 10/8-type conc invts
- adjunction-type sliceness invts

- many of these invarients are easy to compute

<u>Problem</u>: Given K, systematically produce J w/ $\partial X(J) \simeq H_*$ cob to $\partial X(K)$.