Def. A bi-complex C is the following information

$$C = \begin{cases} C_{p,q} & R\text{-modules} \\ \delta_{p,q} : C_{p,q} \to C_{p-1,q} & R\text{-modules homomorphisms.} \\ d_{p,q} : C_{p,q} \to C_{p,q-1} & \end{cases}$$

s.t.

$$\begin{array}{ccccc} & \downarrow & & \downarrow & \\ \leftarrow & C^{p,q+1} & \xleftarrow{\delta} & C^{p+1,q+1} & \leftarrow \\ & \downarrow d & \circlearrowleft & \downarrow d & \\ \leftarrow & C^{p,q} & \xleftarrow{\delta} & C^{p+1,q} & \leftarrow \\ & \downarrow & & \downarrow & \end{array}$$

commutes ; and also $\delta^2=0$; $d^2=0$.

We often assume that $C_{p,q}=0$ if $p<0$ or $q<0$.

__Homologies__ : Since every row & colum are chain complexes,

so $\exists$ homologies $H_{p,q}(C,\delta) = \mathrm{Ker}(\delta_{p,q}) / \mathrm{Im}(\delta_{p+1,q})$.

$H_{p,q}(C,d) = \mathrm{Ker}(d_{p,q}) / \mathrm{Im}(\delta_{p,q+1})$

Define $C_i^{Tot} = \bigoplus_{p+q=i} C_{p,q} = C_{i,0} \oplus C_{i-1,1} \oplus \cdots \oplus C_{0,i}$

Define $D_i : C_i^{Tot} \to C_{i-1}^{Tot}$ by

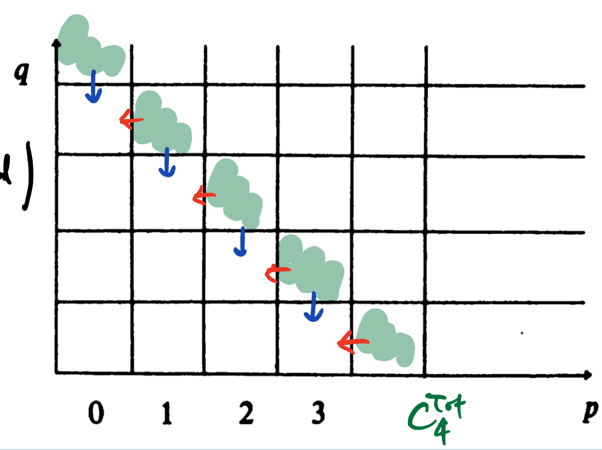
$$D = \bigoplus_{p+q=i} \left(\delta_{p,q} + (-1)^p d_{p,q} \right)$$

__Lemma__: $D \circ D = 0$.

proof : $D \circ D = \left(\delta + (-1)^{p+1} d\right) \circ \left(\delta + (-1)^p d\right)$

$= \delta^2 + (-1)^p d\delta + (-1)^{p+1} \delta d - d^2 = 0.$

$\square$

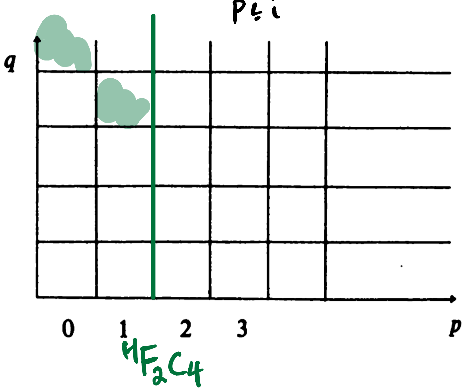


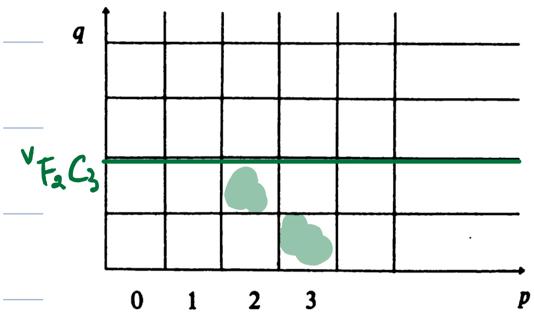
Hence the chain cplx (C_i^{Tot}, D_i^{Tot}) defines a homology $H_i^{Tot}(c) := \ker D_i / \operatorname{Im} D_{i+1}$.

<u>Spectral sequences</u> : Goal: Relates $H_{p,q}(C, \delta)$, $H_{p,q}(C, d)$ & $H_i^{Tot}(C)$.

More <u>definitions</u>: Define the horizontal Filtration & veritical Filtration by

$${}^{H}F_i C_n := \bigoplus_{\substack{p+q=n \\ p \le i}} C_{p,q} \qquad {}^{V}F_i C_n := \bigoplus_{\substack{p+q=n \\ q \le i}} C_{p,q}.$$





For simplicity, write ${}^{H}F_iC_n = F_i C_n$ now. Notice

$D: F_iC_n \to F_iC_{n-1}$ So $F_iC_0 \xleftarrow{D} F_iC_1 \xleftarrow{D} F_iC_2 \xleftarrow{D} \cdots$

is a chain cplx. Write its homology as F_iH_n.

Hence $H_n^{Tot} = F_nH_n$

Let
$$E^r_{p,q} = \frac{F_pC_{p+q} \cap D^{-1}(F_{p-r}C_{p+q-1})}{[F_{p-1}C_{p+q} \cap D^{-1}(F_{p-r}C_{p+q-1})] + [F_pC_{p+q} \cap D(F_{p+r-1}C_{p+q+1})]}$$

Notice, $E^r_{p,q}$ "comes" from $F_pC_{p+q} / F_{p-1}C_{p+q} = C_{p,q}$, i.e. any element in $E^r_{p,q}$ is some quotient of an element in $C_{p,q}$.

$$D(E^r_{p,q}) \subseteq D(F_p C_{p+q}) \cap (F_{p-r} C_{p+q-1}) \;/\sim$$

$$\subseteq \ker D \cap (F_{p-r} C_{p+q-1}) \;/\sim$$

$$\subseteq D^{-1}(F_{p-2r} C_{n-1}) \cap (F_{p-r} C_{p+q-1}) \;/\sim \qquad (\text{since } 0 \in F_{p-2r} C_{n-1})$$

pass to qotient $\big\downarrow$

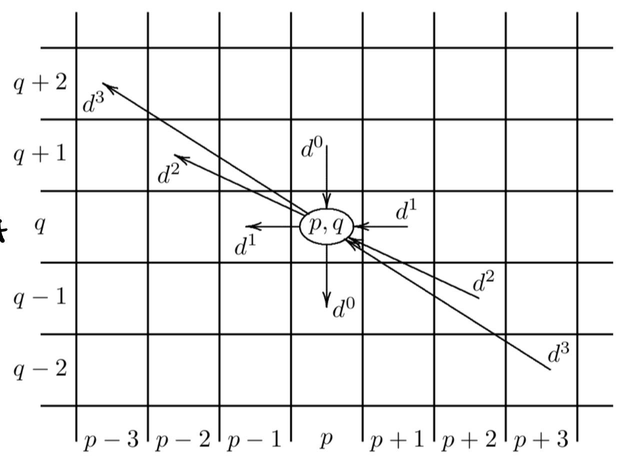
$$\frac{F_{p-r} C_{n-1} \cap D^{-1}(F_{p-2r} C_{n-1})}{[F_{p-2r-1} C_n \cap D^{-1}(F_{p-2r} C_{n-2})] + [F_{p-r} C_{n-1} \cap D(F_{p-1} C_n)]}$$

Hence, we have $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,\, q+r-1}$

<u>Theorem</u>:

- d^r is well-defined
- $E^0_{p,q} = C_{p,q}$; $d^0_{p,q} : C_{p,q} \to C_{p,q-1}$ is the same as $d_{p,q}$
- $E^1_{p,q} = H_{p,q}(C,d)$; $d^1_{p,q} : H_{p,q}(C,d) \to H_{p-1,q}(C,d)$ is induced by $\delta_{p,q}$.
- $E^2_{p,q} = H_{p,q}(H_{*,*}(C,d), \delta)$.
- $E^{r+1}_{p,q} = \operatorname{Ker}(d^r_{p,q}) / \operatorname{Im}(d^r_{p+r,\, q-r+1})$
- When $n > \max(p,q)$, then $E^n_{p,q} = E^{n+1}_{p,q} =: E^\infty_{p,q}$
- $E^\infty_{p,q} = F_p H_{p+q} / F_{p-1} H_{p+q}$.

<u>proof</u>: Homework. //



Let $p+q = n$. Then

$$0 \subseteq F_0 H_n \subseteq F_1 H_n \subseteq F_2 H_n \subseteq \cdots \subseteq F_n H_n = H_n^{Tot}$$

$$E^\infty_{0,n} \qquad E^\infty_{1,n-1} \qquad E^\infty_{2,n-2} \qquad \cdots \qquad E^\infty_{n,0}$$

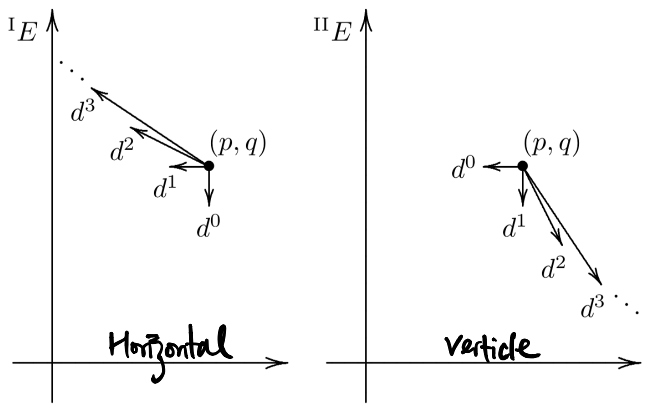
We say $E_{p,q}$ converges to H^{Tot}.

Recall that if $U \subseteq V \subseteq W$ are vector spaces, then

$$\frac{V}{U} \oplus \frac{W}{V} \cong \frac{W}{U}.$$

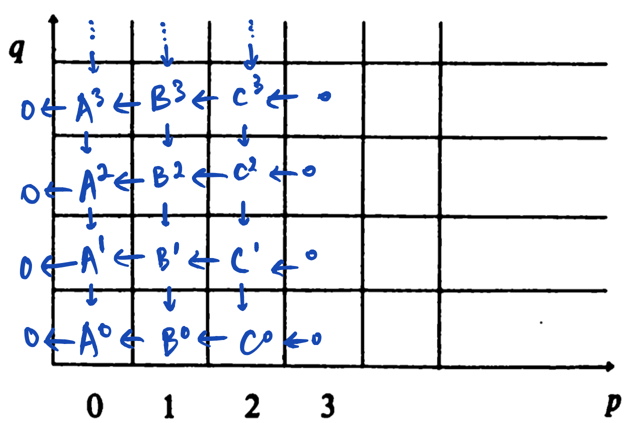
Hence, if all the $C_{p,q}$'s are vector spaces, then $H_n^{tot} = \bigoplus_{p+q=n} E_{p,q}^{\infty}$.

Redo everything with verticle filtration, we have :



Example: Snake lemma. Suppose we have the following

$d_0 \downarrow$ $\quad E^0_{p,q} =$



Such that the rows are exact. Using ${}^{V}F$; we have $E^1_{p,q} = 0 = E^{\infty}_{p,q}$.

Hence $H_n^{Tot} = 0$. Using ${}^{H}F$, we have the following :

$$\overset{d^1}{\leftarrow} \quad E^1_{p,q} = \begin{array}{c|c|c|c|c}
q & \vdots & & & \\
0 \leftarrow & H^3(A) \leftarrow & H^3(B) \leftarrow & H^3(C) \leftarrow & 0 \\
0 \leftarrow & H^2(A) \leftarrow & H^2(B) \leftarrow & H^2(C) \leftarrow & 0 \\
0 \leftarrow & H^1(A) \leftarrow & H^1(B) \leftarrow & H^1(C) \leftarrow & 0 \\
0 \leftarrow & H^0(A) \leftarrow & H^0(B) \leftarrow & H^0(C) \leftarrow & 0 \\
\hline
 & 0 & 1 & 2 & 3 \quad p
\end{array}$$

$$\overset{d^2}{\nwarrow} \quad E^2_{p,q} = \begin{array}{c|c|c|c|c}
q & \vdots & \vdots & \vdots & \\
3 & \operatorname{Coker} d^1_{1,3} & \dfrac{\ker d^1_{1,3}}{\operatorname{Im} d^1_{2,3}} & \ker d^1_{2,3} & 0 \\
2 & \operatorname{Coker} d^1_{1,2} & \dfrac{\ker d^1_{1,2}}{\operatorname{Im} d^1_{2,2}} & \ker d^1_{2,2} & 0 \\
1 & \operatorname{Coker} d^1_{1,1} & \dfrac{\ker d^1_{1,1}}{\operatorname{Im} d^1_{2,1}} & \ker d^1_{2,1} & 0 \\
0 & \operatorname{Coker} d^1_{1,0} & \dfrac{\ker d^1_{1,0}}{\operatorname{Im} d^1_{2,0}} & \ker d^1_{2,0} & 0 \\
\hline
 & 0 & 1 & 2 & 3 \quad p
\end{array}$$

Since the second column is stabalize, and $E^\infty_{p,q} = 0$, so the second col. is zero.

Since the first & third column stabalize at E^3, so the red arrow d^2 are isomorphism.

Hence, we have a long exact sequence

$$\to H^n(C) \xrightarrow{d^1} H^n(B) \xrightarrow{d^1} H^n(A) \xrightarrow{\text{connection homomorphism}} H^{n-1}(C) \xrightarrow{d^1} H^{n-1}(B) \xrightarrow{d^1} \cdots$$

$$\searrow^{d^1} \operatorname{coker} d^1_{1,n} \overset{d^2}{=} \ker d^1_{2,n-1} \nearrow^{d^1}$$

Theorem (Leray - Serre)

let $F \hookrightarrow X \to B$ be a fibration (e.g. a fibre bundle), $\pi_1(B)=0$. Then $\exists$ a spectral sequence $E_{p,q}$ with horizontal filtration, such that it converges to $H_\bullet(X)$, and $E^2_{p,q} = H_p(B; H_q(F))$.

Example : Computation of $H_\bullet(\mathbb{C}P^n)$. Let $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration. Then

$$E^2_{p,q} = \begin{array}{c|cccc}
q & \vdots & \vdots & \vdots & \\
 & H_0(\mathbb{C}P^n; H_2 S^1) & H_1(\mathbb{C}P^n; H_2 S^1) & H_2(\mathbb{C}P^n; H_2 S^1) & \cdots \\
 & H_0(\mathbb{C}P^n; H_1 S^1) & H_1(\mathbb{C}P^n; H_1 S^1) & H_2(\mathbb{C}P^n; H_1 S^1) & \cdots \\
 & H_0(\mathbb{C}P^n; H_0 S^1) & H_1(\mathbb{C}P^n; H_0 S^1) & H_2(\mathbb{C}P^n; H_0 S^1) & \cdots \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

$$= \begin{array}{c|cccc}
q & & \text{all zero} & & \\
1 & H_0(\mathbb{C}P^n; \mathbb{Z}) & H_1(\mathbb{C}P^n; \mathbb{Z}) & H_2(\mathbb{C}P^n; \mathbb{Z}) & \cdots \\
0 & H_0(\mathbb{C}P^n; \mathbb{Z}) & H_1(\mathbb{C}P^n; \mathbb{Z}) & H_2(\mathbb{C}P^n; \mathbb{Z}) & \cdots \\
\hline
 & 0 & 1 & 2 & \cdots \quad p
\end{array}$$

$d^2 \qquad d^2$

Since the only non-trivial d^r is the d^2, so

$E^3_{p,q} = E^4_{p,q} = \cdots = E^\infty_{p,q}$, converging to $H_{p+q}(S^{2n+1})$

Hence, $H_0(\mathbb{C}P^n; \mathbb{Z}) = E^\infty_{0,0} = H_0(S^{2n+1}) = \mathbb{Z}$

$H_1(\mathbb{C}P^n; \mathbb{Z}) = E^\infty_{1,0} \subseteq H_1(S^{2n+1}) = 0$.

Since $H_i(S^{2n+1})=0$ except when $i=2n+1$

$\rightsquigarrow$ $E^\infty_{p,q}=0$ except when $p+q=2n+1$.

$\therefore$ Most of the d^2 are isomorphism (when $p+q=2n+1$)

$\rightsquigarrow$ $H_{2i-1}(\mathbb{C}P^n;\mathbb{Z})=0$ and $H_{2i}(\mathbb{C}P^n;\mathbb{Z})=\mathbb{Z}$. $i\le n$.

and $H_i(\mathbb{C}P^n;\mathbb{Z})=0$ $i>2n+1$

When $p+q=2n$:

$$E^2_{p,q}=\begin{array}{|cccccc}
\cdots & H_{2n-1}(\mathbb{C}P^n)=0 & H_{2n}(\mathbb{C}P^n)=\mathbb{Z} & H_{2n+1}(\mathbb{C}P^n) & 0 & \cdots\ 0 \\
\cdots & H_{2n-1}(\mathbb{C}P^n)=0 & H_{2n}(\mathbb{C}P^n)=\mathbb{Z} & H_{2n+1}(\mathbb{C}P^n) & 0 & \cdots\ 0 \\
\hline
\end{array}$$

From the picture, $E^2_{2n,1}$ stabalizes. So $E^\infty_{2n,1}=E^2_{2n,1}=\mathbb{Z}$.

Hence $0=F_0H_{2n+1}=\cdots=F_{2n-1}H_{2n+1}\underbrace{\subseteq}_{E_{2n,1}=\mathbb{Z}}F_{2n}H_{2n+1}\underbrace{\subseteq}_{E_{2n+1,0}}H_{2n+1}=\mathbb{Z}$

$\rightsquigarrow$ $E^\infty_{2n+1,0}=0$. So

$$H_{2n-1}(\mathbb{C}P^n)=0 \longleftarrow H_{2n+1}(\mathbb{C}P^n)$$

is an isomorphism $\rightsquigarrow$ $H_{2n+1}(\mathbb{C}P^n)=0$.

$$\therefore\ H_i(\mathbb{C}P^n)=\begin{cases}0 & i \text{ odd or } i>2n\\ \mathbb{Z} & i \text{ even and } 0\le i\le 2n\end{cases}$$

Sketch proof of the theorem: Suppose $F \hookrightarrow X \xrightarrow{p} B$ is a fibre bundle. Let $\{U_\alpha\}$ be a good cover of B, i.e. any finite intersection of U_α is either empty or contractible.

By choosing a finer good cover, suppose $p^{-1}(U_\alpha) \cong U_\alpha \times F$.

Let $C_{p,q} = \check{C}_p\left(\{U_\alpha\}; C_q^{sing}(p^{-1}(U_\alpha))\right)$

By Mayer-Vietoris, if we use verticle filtration, we have

$$E^1_{p,q} = \begin{cases} 0 & \text{if } p \neq 0 \\ C_q^{sing}(B) & \text{if } p = 0. \end{cases}$$

Hence, $E_{p,q}$ converges to $H_\bullet(B)$. Then, using horizontal filtration, we have

$$E^1_{p,q} = \check{C}_p(\{U_\alpha\}; H_q(F))$$

$$E^2_{p,q} = \check{H}_p(\{U_\alpha\}; H_q(F))$$
$$= H_p(B; H_q(F)).$$

The coefficient $H_q(F)$ might be twisted $\leadsto$ requires $\pi_1(B) = 0$.

$\square$.