Spinelessness---Hayden-Piccirillo

Thursday, May 28, 2020 12:06 PM

Theorem: There exist smooth 4-manifolds W and W' such that

(1) W and W' are homeomorphic

(2) W' is diffeomorphic to the 0-trace of a slice knot

(3) There is no smooth embedding of any knot trace into W which induces an isomorphism on homology

($\Rightarrow$ W is not diffeomorphic to any knot trace).

The trace embedding lemma and spinelessness

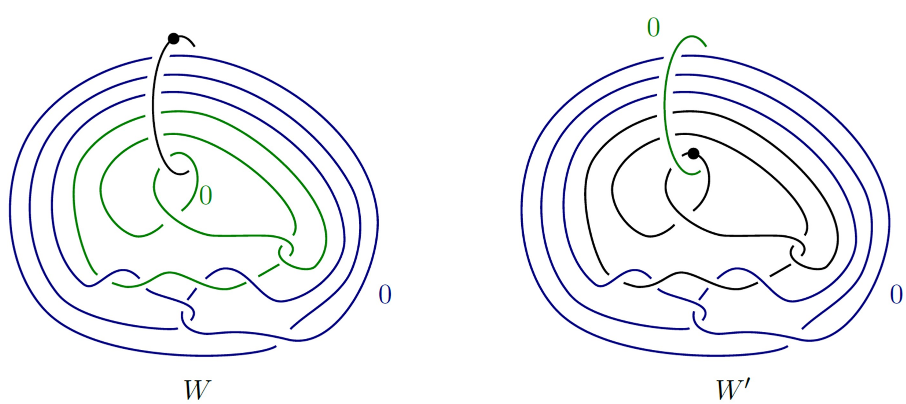


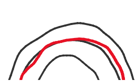
Figure 2: Diagrams for W and W' related by a dot-zero exchange.

Spine: A (top loc flat/smooth/PL) spine for a manifold W^4 is an (top locflat/smooth/PL) embedding

$$f: \underset{\text{manifold}}{\Sigma^2} \longrightarrow W^4$$

such that the embedding induces a homotopy equivalence.

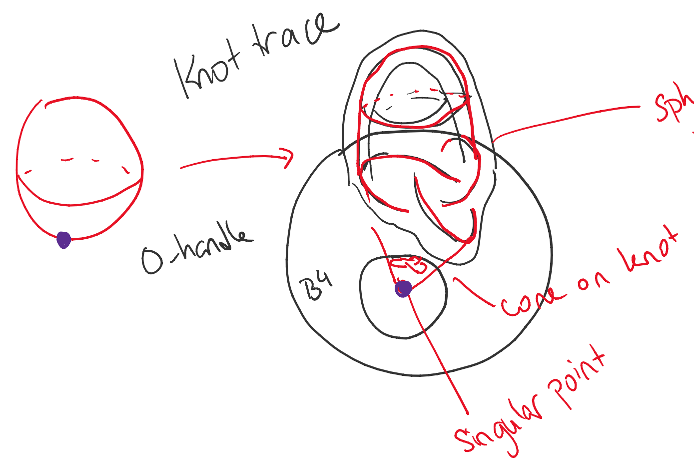
Example: Any knot trace has a PL 2-sphere spine

Knot trace  not smoothly embedded

...oothly embedded

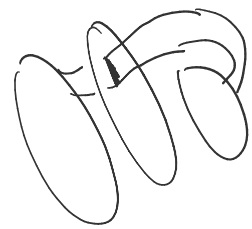
R^4

Knot trace

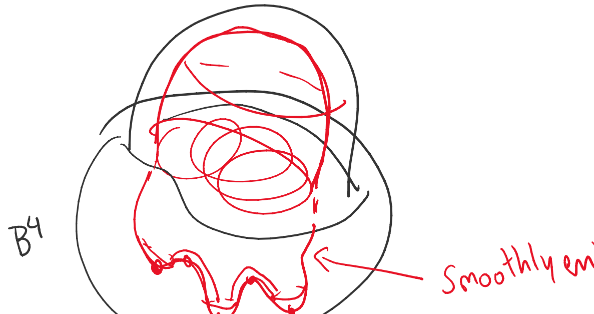


Homotopy type of any knot trace is equivalent to S^2

If the knot is slice, then have a smoothly embedded spine



Ribbon knots are slice
Can get a smoothly embedded disk in B^4 with boundary on knot



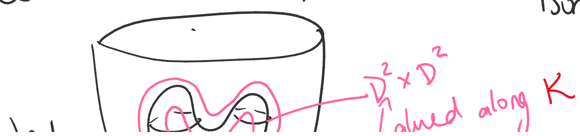
<u>Corollary of Theorem</u>: W (not a knot trace) is homotopy equivalent to S^2 and contains a topologically locally flat spine but not a <u>smooth</u> or PL spine.

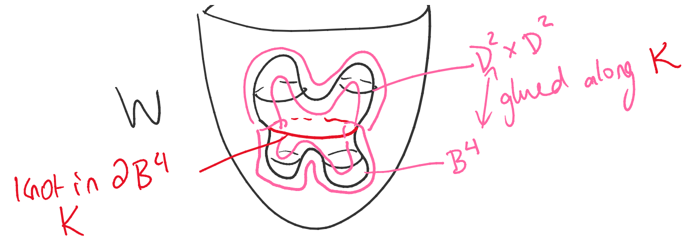
Theorem says W is homeomorphic to W' and W' is diffeomorphic to 0-trace on a slice knot $\Rightarrow$

$$\underset{\text{smooth spine}}{S^2 \hookrightarrow} W' \xrightarrow{\text{homeomorphism}} W$$

$\Rightarrow$ W contains a topologically locally flat spine

If W had a smooth spine then a nbhd of that spine is a knot trace
inclusion of knot trace into W induce isomorphism on homology
contradicts

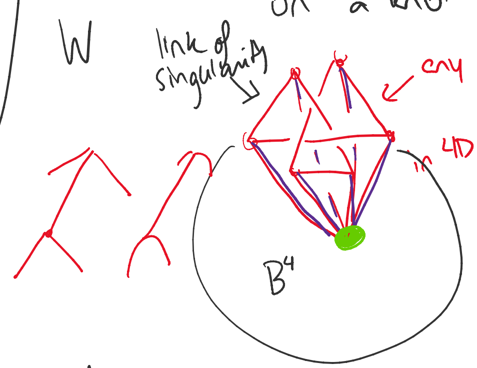




contradicts Theorem

PL version: also find that a nbhd of PL spine will be a knot trace

At vertices might see cone on a knot



The rest of S^2 can be smoothed to a smooth embedding of D^2 ← its nbhd gives a 2-handle attached to B^4 nbhd of tree ← attaching sphere: connected sum of the links of vertices

→ embedding of knot trace

Thm: W, W' ← diffeo to knot trace. W homeo to W'

There is no (smooth) embedding of any knot trace into W inducing an isomorphism on homology.

Proof outline:

(1) W embeds into S^4 ← add 2-handle, to diagram 3-h, 4-h do Kirby calculus to show result of adding handles is diffeo to S^4

(2) W satisfies the adjunction inequality:

$$[\Sigma]\cdot[\Sigma] + |\langle c_1(W), [\Sigma]\rangle| \leq 2g(\Sigma) - 2$$

for any surface Σ representing a nontrivial homology class.

W has nice presentation in contact/symplectic topology (Stein)
(Use gauge theory, symplectic, complex

for any surface Σ representing a nontrivial homology class. (Use gauge theory, symplectic, complex geometry)

③ W and W' are homeomorphic

Mazur cork twist

④ W' is diffeomorphic to a knot trace ← Kirby calculus simplify diagram by cancelling 1-handle + 2-handle

Using trace embedding lemma, if there were any embedding of a knot trace

$$\underline{X_n(K) \hookrightarrow W}$$ inducing an iso on homology

then $X_n(K) \hookrightarrow W \hookrightarrow S^4 \implies n=0$ and K is slice

$\implies$ there is a smooth 2-sphere Σ in $X_n(K)$ generating $H_2(X_n(K))$

$$\Sigma \hookrightarrow X_n(K) \hookrightarrow W$$

Σ represents a nontrivial homology class in W

$$\implies \underbrace{[\Sigma]\cdot[\Sigma]}_{0} + |\langle c_1(W), [\Sigma]\rangle| \leq 2g(\Sigma)-2$$

$$0 + |\langle c_1(W), [\Sigma]\rangle| \leq -2$$

$$0 \leq -2 \implies \Leftarrow$$ contradiction

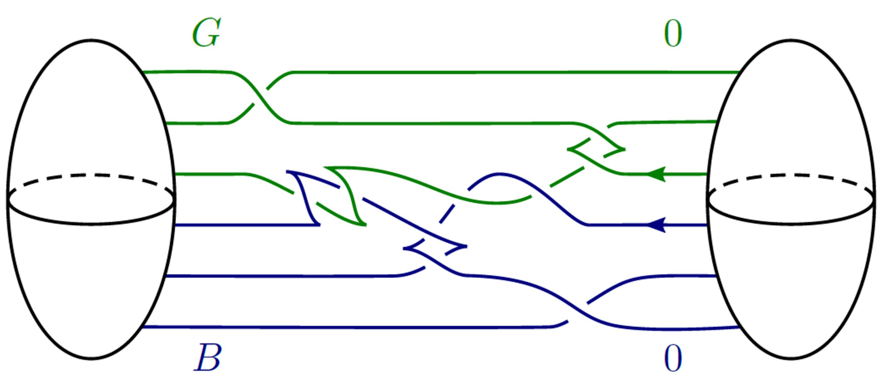


Figure 1: A Stein handlebody diagram for W.