Movies

Thursday, April 16, 2020 2:01 PM

(1) Recap on s-invariant

K = knot $\Rightarrow$ spectral sequence

$$H_{Kh}(K) \longrightarrow H_{Lee}(K)$$

bigraded by q, t-degrees — t-graded q-filtered.

x = generator in Lee homology

$s(x)$ = maximal q-grading of its lifts to Khovanov

$$s_{max} = \max\{s(x) : x \underset{\neq 0}{\in} H_{Lee}\}$$

$$s_{min} = \min\{s(x) : x \neq 0 \text{ in } H_{Lee}\}$$

Last time: We proved $s_{max} \not\equiv s_{min} \bmod 4$.

Lee's canonical generators z_1, z_2

$$\Rightarrow s_{max} = s(z_1) = s(z_2)$$

$$s_{min} = s(z_1 \pm z_2)$$

Thm (Rasmussen) $s_{max} = s_{min} + 2$.

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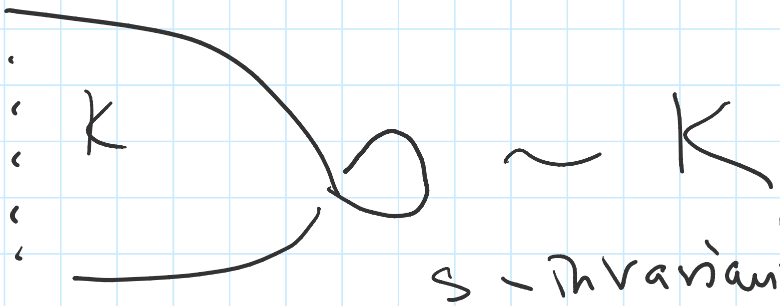
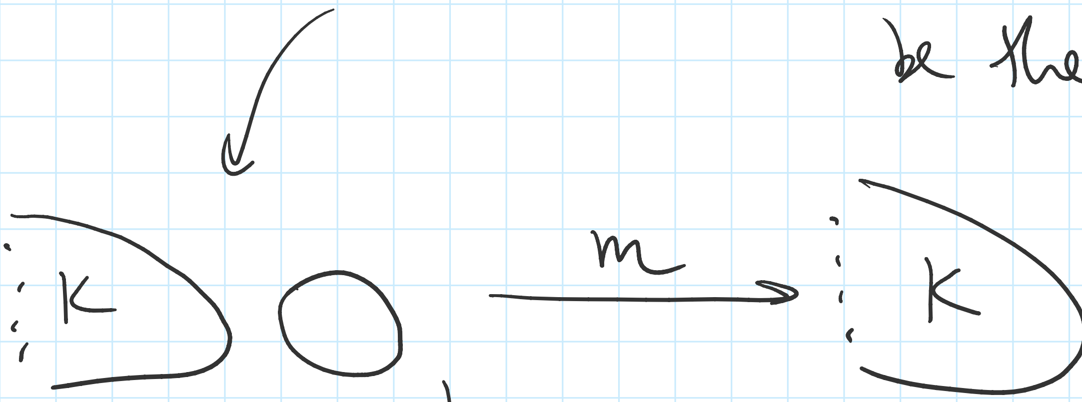
Idea



s-invariant should be the same.



$(z_1 \pm z_2)$ $\quad a, b \quad$ $a = 1+x \quad b = 1-x$

Suppose z_1 has a near the crossing.
z_2 has b near the crossing.

Know $a \cdot a = (1+x)(1+x) = 2 + 2x = 2a$
$a \cdot b = 0$ in Lee homology.

$$(z_1 \pm z_2) \cdot a = 2 z_1$$

has filtration s_{min} $\quad$ has filtration 2 $\quad$ has filtration s_{max}

$$\Rightarrow s_{min} + 2 \geq s_{max}$$

$s_{min} < s_{max}$

Now s_{min}, s_{max} have same parity, different remainders mod 4 $\Rightarrow$ $s_{min} < s_{max}$.

$$\boxed{s_{min} = s_{max} - 2.}$$

Def $s(K) = s_{max} - 1 = s_{min} + 1$

Ex K is a positive knot (all crossings are positive).

$\Rightarrow$ oriented resolution is the same as 0-resolution, on the left end of the cube.

Suppose that in this resolution we have n crossings and k circles.

z_1, z_2 = some product of a's and b's with k factors

$\deg_A = 2k$

$\Rightarrow$ top degree term $= \underbrace{x \otimes \ldots \otimes x}_{k}$

$\Rightarrow s_{max} = 2k - \#\text{circles} - |v| - n_+ + 2n_-$

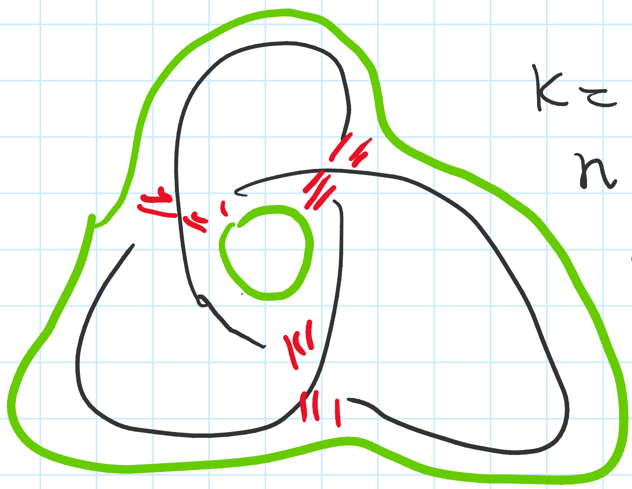
$= 2k - k - 0 - n + 0$

$= 2k - k - 0 - n + 0$

$= k - n$

$\Rightarrow$ Know s-invariant!

$$\boxed{s(K) = k - n - 1}$$



$k = 2$

$n = 3$

$s = 2 - 3 - 1 = -2$

Can also consider Seifert surface by filling circles with disks and gluing them with twisted bands

$\chi(S) = k - n = 2 - 2g(S) - 1$

one bdry component

$$\boxed{\underset{s(K)}{\underset{\parallel}{k-n-1}} = -2g(S)}$$

Conclusion: If K is a positive knot

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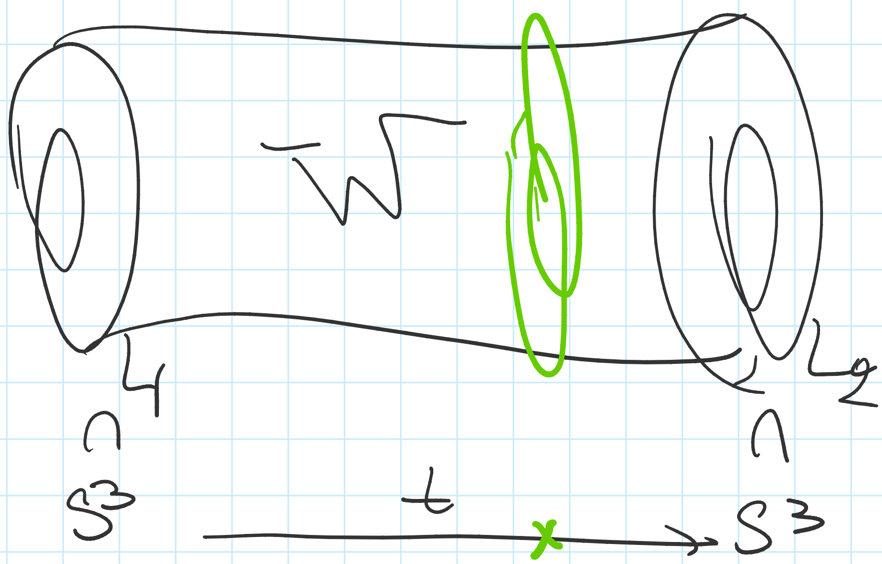
then $\boxed{s(K) = -2g(\text{Seifert surface obtained by Seifert algorithm})}$.

Next time $|s(K)| \leq 2g_4(K) \leq 2g(K)$

② Link cobordisms and movies.



$W \subset S^3 \times [0,1]$ smooth surface such that $W|_{t=0} = L_1$ $W|_{t=1} = L_2$

$\partial W = L_1 \cup L_2$

For links, we use diagrams and Reidmeister moves.

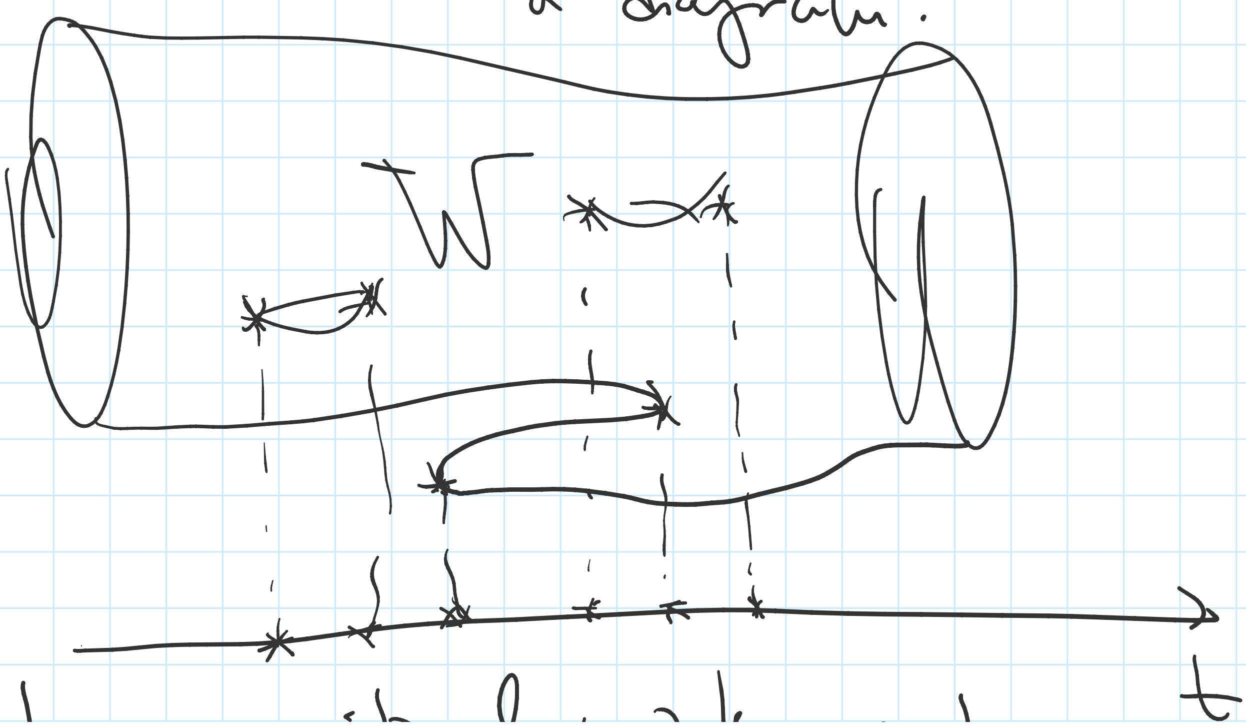
Reidmeister moves.

Q: Can we do the same for surfaces?

Idea: Put W in general position and regard t as a (Morse) function on W.

$\Rightarrow$ for each $t \in [0,1]$ we can consider the "slice" $W_t \subset S^3$; for all but finitely many t this is a smooth link in S^3

$\rightsquigarrow$ can present it by a diagram.



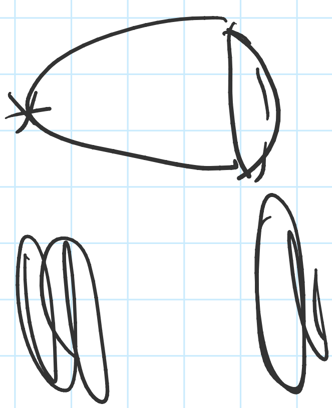
t

between critical points, we have smooth links in S^3, and at critical points we have one of 3 cases:

(a) Maximum (local) of t

$\Longleftrightarrow$ one link component shrinks and disappears.

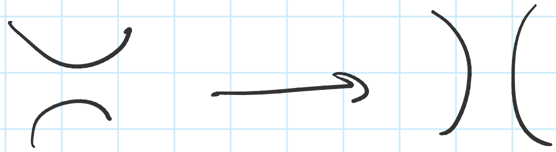
(b)



Minimum of t

$\Rightarrow$ create a small circle

(c)



saddle

this happens in some link diagram.

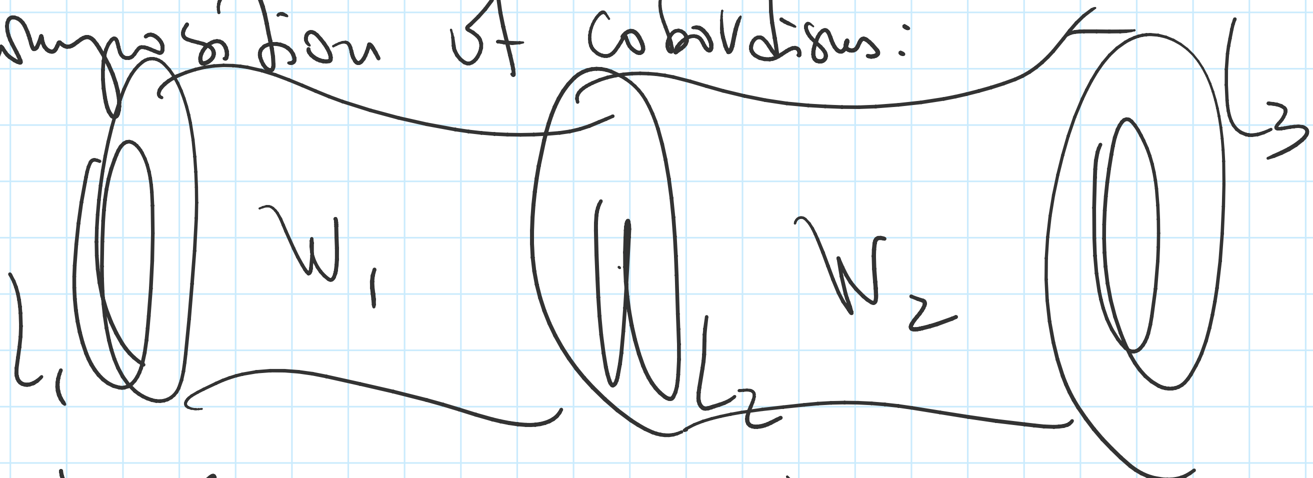
(d) Also between critical points

(d) Also, between critical points we need to consider isotopies of links in S^3 $\longleftrightarrow$ Reidemeister moves.

Conclusion Any such cobordism W can be presented as a "movie" of link diagrams.

Each "frame" is a link diagram. Next "frame" differs from a previous one by Reidemeister move or a Morse move (= create/kill a circle, saddle)

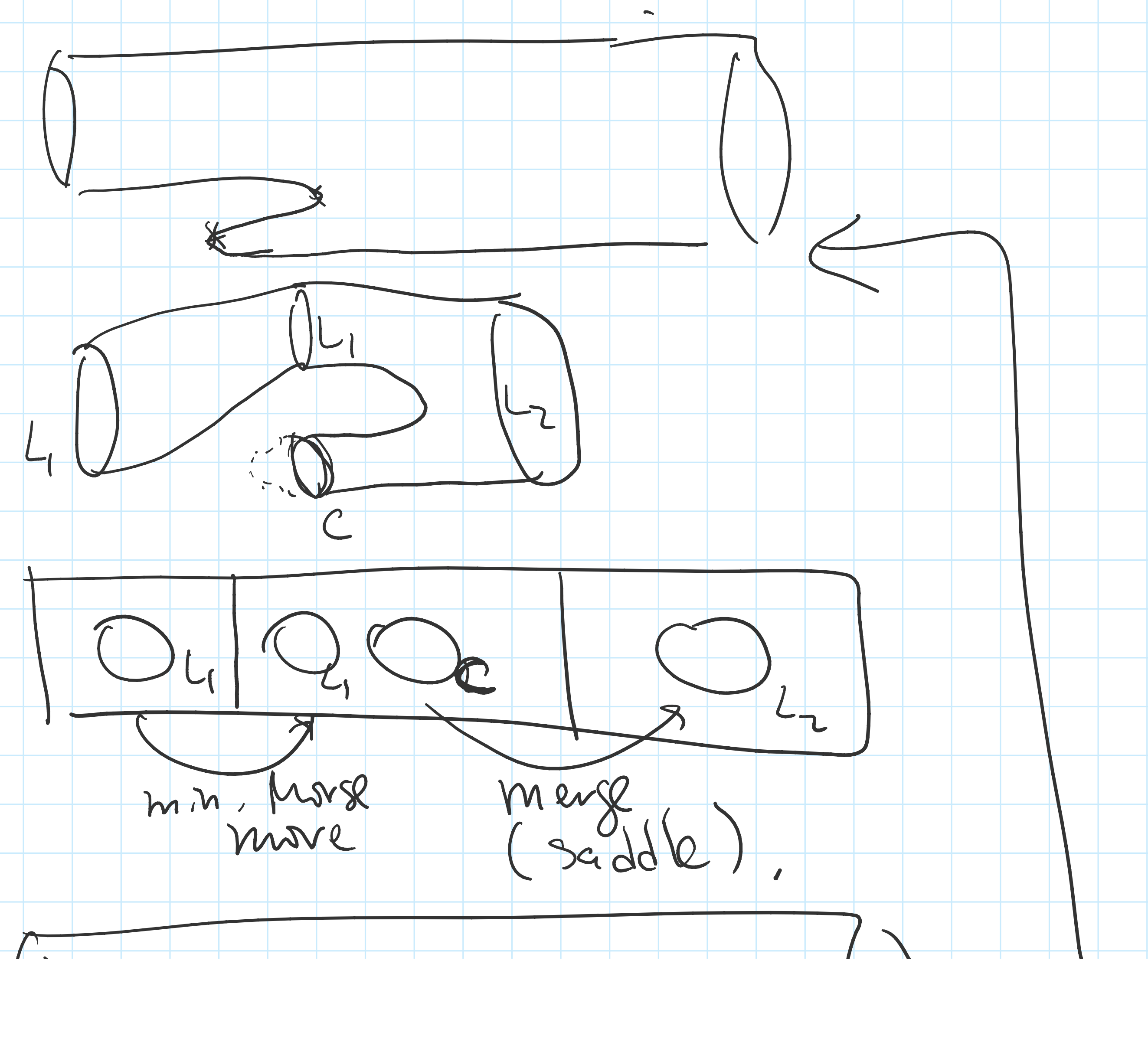
Composition of cobordisms:

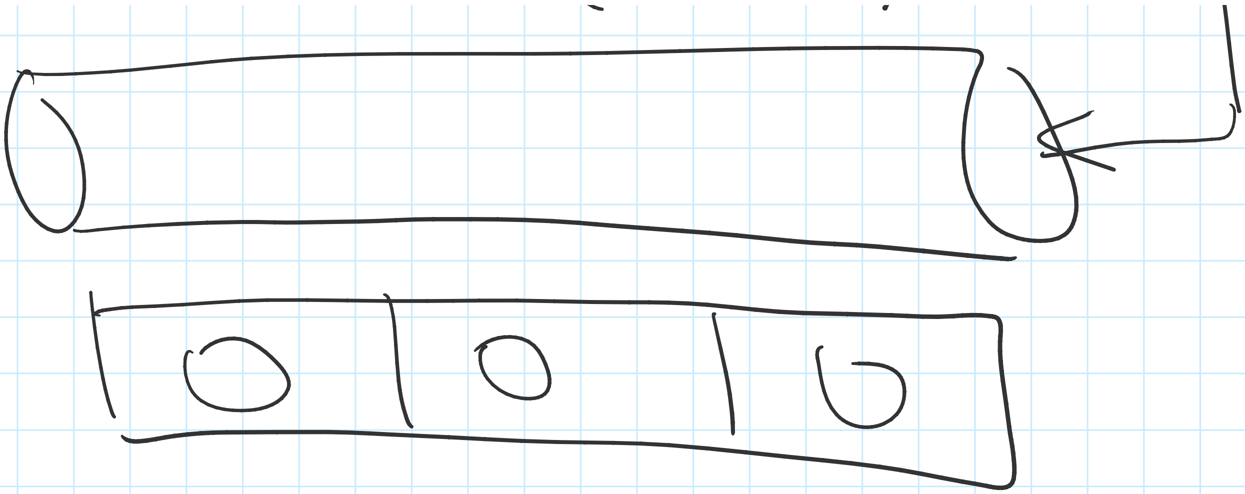


Note The same cobordism can

__Note__ The same cobordism can be presented by different movies

__Carter–Saito__: Two movies represent the same cobordism (isotopic in $S^3 \times [0,1]$) $\iff$ they can be obtained from each other by a sequence of 'movie moves'





Thm [Khovanov] W = cobordism between L_1 and L_2
$\Rightarrow$ there is a linear map
$$H_{Kh}(L_1) \xrightarrow{\Phi_W} H_{Kh}(L_2)$$

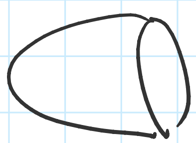
Thm 2 [Jacobsson, Blanchet, Caprau, ...] Φ_W is invariant under isotopies of W (up to sign)

Idea: Present W as a sequence of frames.

Need to define Φ_W for each Reidemeister or Morse move and compose all these.

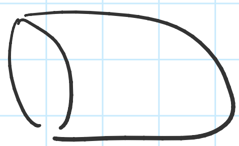
and compose all these.

Reidemeister moves $\longleftrightarrow$ follows from invariance of H_{Kh}



$$\mathbb{R} \xrightarrow{i} A$$

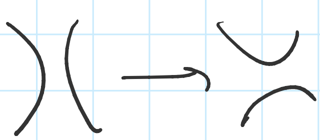
$$1 \longrightarrow 1$$



$$A \xrightarrow{\varepsilon} R$$

$$\varepsilon(x) = 1$$

$$\varepsilon(1) = 0$$



m / M.

<u>Idea of proof of Th 2:</u> Check that

Φ_W defined this way is preserved by all movie moves!